Geometry of Manifolds

Lectures delivered by Tobias Colding
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Introduction

Tobias Colding taught a course (18.965) on Geometry of Manifolds at MIT in Fall 2012. These are my "live-TeXed" notes from the course. The template is borrowed from Akhil Mathew.

Please email corrections to holden1@mit.edu. Thanks to Fan Zheng for corrections.

Lectures 23–24 are unedited.

Lecture 1
Thu. 9/6/12

Today Bill Minicozzi (2-347) is filling in for Toby Colding.

We will follow the textbook Riemannian Geometry by Do Carmo. You have to spend a lot of time on basics about manifolds, tensors, etc. and prerequisites like differential topology before you get to the interesting topics in geometry. Do Carmo gets to the interesting topics much faster than other books.

Today we give a quick overview of Riemannian geometry, and then introduce the basic definitions (manifolds, tangent spaces, etc.) that we'll need throughout the course. You will see how these definitions generalize concepts you are already familiar with from calculus.

§1 What is Riemannian geometry?

On Euclidean space we can do calculus; we can measure distances, angles, volumes, etc.

However, we want to do all that geometry on more general spaces, called **Riemannian manifolds**.

First, we'll have to rigorously define what those spaces are. What is a manifold? We need to generalize the basic notions of calculus in the manifold setting: what is a derivative? A derivative is basically a linear approximation, because the tangent line is the best linear approximation. We'll define the notion of a **tangent space** for a manifold.

Once we have a manifold, we can define have functions, curves and (sub)surfaces on the manifolds, and objects called **tensors**. The idea of tensors generalizes the idea of vector fields, which are 1-tensors. We can differentiate tensors; for instance, the covariant derivative of two-tensor is three-tensor.

Next, we'll see that a **Riemannian metric** allows us to calculuate distance and angles. A **geodesic** is the shortest path connecting two points, or more generally, paths that are *locally* shortest. For instance, the equator of a sphere is a geodesic: any connected part of the diameter that doesn't include antipodal points gives the shortest path between two points. We can view our spaces as metric spaces and do some geometry. We have *comparison theorems*, where we use the geometry of the space to get information about the metric. For instance, in the Bonnie-Meyer theorem, we use the curvature of a space to learn about its metric.

Later in the course, we will cover topic such as Cartan-Hadamard manifolds, harmonic maps, and minimal surfaces.

§2 Manifolds

We want to do calculus on more general spaces, called manifolds. In particular, we care about (smooth) differential manifolds. Before we give a formal definition, we first develop some intuition through examples.

2.1 Examples

The following are all manifolds.

- $\mathbb{R}^n$: n-dimensional Euclidean space

$$\mathbb{R}^n := \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}.$$

- S^n: the unit n-sphere

$$S^n := \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}.$$

  Here the Euclidean norm is defined by $|x|^2 := \sum_{i=1}^n x_i^2$. Note this is an example of a "submanifold" of $\mathbb{R}^{n+1}$.

Note that Euclidean geometry descends to geometry on any submanifold. A theorem of Nash says any abstract manifold can be embedded (at least locally) in Euclidean space. This means it is sufficient to learn about geometry of submanifolds of Euclidean space.

However, just as linear algebra is often simpler with "linear transformations" than with matrices, we will see that geometry is often simpler when we think of manifolds in the abstract.

Here are some more examples.

- T^n: n-torus $\mathbb{R}^n/\mathbb{Z}^n$. This means that we are modding out $\mathbb{R}^n$ by the equivalence relation $\sim$ where $x \sim (x+z)$ for every tuple $z = (z_1, \ldots, z_n)$ with $z_i \in \mathbb{Z}$. Note any small piece of T^n looks like $\mathbb{R}^n$ because don't see the wraparound.

  This local property means we can calculate derivatives of a function defined on T^n the same way we calculate derivatives of a function on $\mathbb{R}^n$.

- $\mathbb{R}P^n$: real projective n-space, the space of lines through 0 in $\mathbb{R}^{n+1}$. Note $\mathbb{R}P^n$ is closely related to the S^n, as follows. Each line through origin cuts sphere in 2 points, so we can think of $\mathbb{R}P^n$ as S^n modded out by the antipodal map $p \mapsto -p$.

These are all differential manifolds, but we don't get a *geometry* on them until we get a Riemannian metric (something we'll develop later in the course).

We also need a notion of a tangent vector. We'll give a formal definition of a manifold, then go back to talk about tangent spaces on manifolds.

2.2 Formal definition

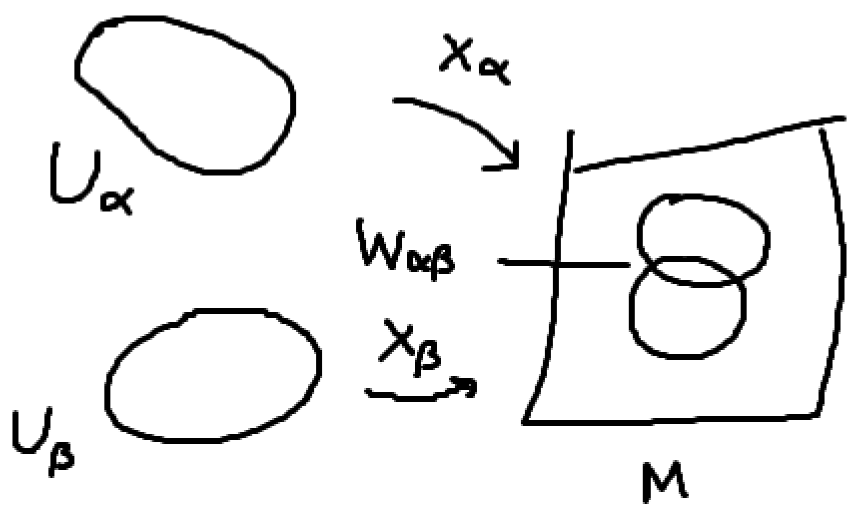
Definition 1.1: A (smooth) n-dimensional **manifold** M (also written M^n) is...

1. a set, denoted M, equipped with

2. a family of open sets $U_\alpha \subseteq \mathbb{R}^n$ and injective maps $x_\alpha : U_\alpha \hookrightarrow M$ (together called a **chart**) such that

   - (The open sets cover the manifold) $\bigcup_\alpha x_\alpha(U_\alpha) = M$.

- (Consistency of overlaps) Set $W_{\alpha\beta} := x_\alpha(U_\alpha) \cap x_\beta(U_\beta)$ for each α and β. Suppose $W_{\alpha\beta} \neq \phi$. Then
  (a) (Topological condition) $x_\alpha^{-1}(W_{\alpha\beta})$ is open,
  (b) (The maps between the subsets of $\mathbb{R}^n$ are smooth) The maps $x_\beta^{-1} \circ x_\alpha$ are C^∞, i.e. infinitely differentiable.
  (c) (* Technical condition) This family is maximal with respect to A and B.



Let's analyze this definition. The open sets U_α tell us that locally, each point is parameterized by an open set in Euclidean space. We saw this in each of the examples. (For S^n, you can "flatten" any local part of the sphere.) Note that M inherits a topology by deeming that each $x_\alpha(U_\alpha)$ be a homeomorphism onto an open set of M.

The technical overlap properties force the x_α to be nice maps. (b) is why we call the manifold "smooth." We can loosen, tighten, or change the condition, for instance,

- A real analytic manifold is where $x_\beta^{-1} \circ x_\alpha$ are all real analytic. (Stricter condition)
- A C^n manifold is one where $x_\beta^{-1} \circ x_\alpha$ are all C^n (n times continuously differentiable). (Looser condition)
- A complex manifold is one where we replace $\mathbb{R}$ with $\mathbb{C}$ and C^n by holomorphic.
- A PL manifold is where $x_\beta^{-1} \circ x_\alpha$ are all piecewise linear.

Without condition (c), we would have a lot of manifolds. Suppose we have $(M, \{x_\alpha\}, \{U_\alpha)$ satisfying all the conditions except (c). For each U_α, we can take a subset $V \subseteq U_\alpha$ and restrict x_α to V. This is still a good parameterization. Adding V and $x_\alpha|_V$, we get a new manifold.

Thus (c) gives uniqueness: two manifolds that should be the same *are* the same.

An alternative approach is as follows: call something satisfying just (a) and (b) quasi-manifolds. Define an equivalence relation: two manifolds are the same if you can refine the two families of mappings $\{(U_\alpha, x_\alpha)\}$ to be the same family. Modding out quasi-manifolds by this equivalence relation gives a manifold.

Note n has to be constant: A sphere with two 1-dimensional antlers is not a manifold.

2.3 Reconciling definition with example

Let's show that $\mathbb{R}P^n$ is a manifold. Define **homogeneous coordinates** as follows: Consider $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \text{at least one } x_i \neq 0\}$, and mod out by the equivalence relation

$$(x_1, \ldots, x_{n+1}) \sim \lambda(x_1, \ldots, x_{n+1})$$

where $\lambda \in \mathbb{R}\backslash\{0\}$. Let the $[x_1, \ldots, x_{n+1}]$ denote the equivalence class of $(x_1, \ldots, x_{n+1})$; it is called homogeneous coordinates.

<u>Defining the open sets and maps:</u>

Define sets $V_i = \{[x_1, \ldots, x_{n+1}] : x_i \neq 0\}$. It's clear that $\bigcup V_i = \mathbb{R}P^n$. Because we need to get $n+1$ coordinates out of n coordinates, we define maps $x_i : \mathbb{R}^n \to V_i$ by

$$x_i(y_1, \ldots, y_n) = [y_1, \ldots, \underbrace{1}_{i}, \ldots, y_n].$$

For instance, for $\mathbb{R}P^3$, we have

$$\begin{aligned} x_1(y_1, y_2) &= [1, y_1, y_2] \\ x_2(y_1, y_2) &= [y_1, 1, y_2] \\ x_3(y_1, y_2) &= [y_1, y_2, 1] \end{aligned}$$

Note these maps are all bijective. They are onto because any element of $[x_1, \ldots, x_n] \in V_i$ has $x_i \neq 0$, and $[x_1, \ldots, x_n] = \left[\frac{x_1}{x_i}, \ldots, \frac{x_i}{x_i} = 1, \ldots, \frac{x_n}{x_i}\right]$.

<u>Verifying overlap properties:</u>

(a) We have
$$W_{12} = x_1(\mathbb{R}^2) \cap x_2(\mathbb{R}^2) = V_1 \cap V_2 = \{[z_1, z_2, z_3] : z_1 z_2 \neq 0\}.$$
Consider $x_1^{-1}(W_{12})$. We have $[z_1, z_2, z_3] \sim \left[1, \frac{z_2}{z_1}, \frac{z_3}{z_1}\right]$, so
$$x_1^{-1} W_{12} = \{(y_1, y_2) : y_1 \neq 0\}.$$
which is open.

(b) Now consider $x_2^{-1} \circ x_1 : W_{12} \to W_{21}$. We have for $y_1 \neq 0$ that
$$(y_1, y_2) \xrightarrow{x_1} [1, y_1, y_2] = \left[\frac{1}{y_1}, 1, \frac{y_2}{y_1}\right] \xrightarrow{x_2^{-1}} \left(\frac{1}{y_1}, \frac{y_2}{y_1}\right).$$
This is rational, so smooth.

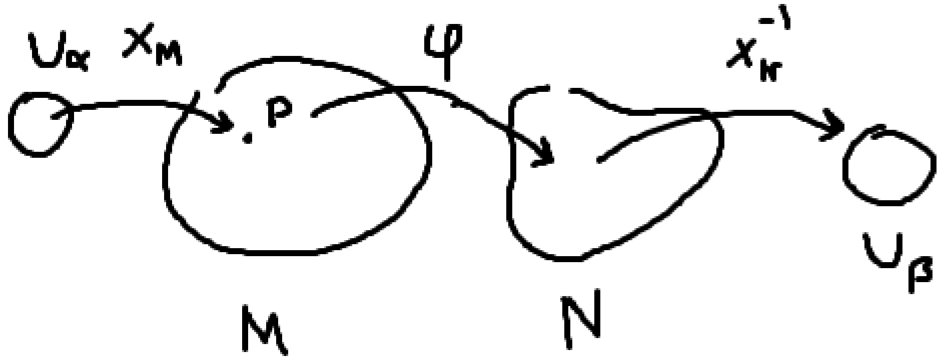
(c) To satisfy condition (c), we take a maximal family of (V_α, x_α) satisfying (a) and (b) and containing all the (V_i, x_i). (I.e. take all intersections among all the (V_i, x_i) and add them in; now take all subsects, not take intersections again, *ad infinitum.*)

Because the calculations are straightforward, this is the first and last time we're going to check something is a manifold.

2.4 Maps between manifolds

Definition 1.2: Let M and N be smooth manifolds. We say that $\varphi : M \to N$ is smooth at $p \in M$ if $x_N^{-1} \circ \varphi \circ x_M$ is smooth at $x_M^{-1}(p)$.

Here, x_M is any x_α such that $p \in x_\alpha(U_\alpha)$, and x_N is any x_β such that $\varphi(p) \in x_\beta(U_\beta)$.



Note the choice of $x_M = x_\alpha$ and $x_N = x_\beta$ doesn't matter, because the transition condition will give that it is true for any choice.

Some particularly important smooth maps are those with domain or target inside $\mathbb{R}$:

- Smooth functions on M, i.e. smooth maps $M \to \mathbb{R}$. This set is denoted by $\mathcal{D}$.
- Curves, maps from an interval $I \subseteq \mathbb{R} \to M$.

§3 Tangent vectors

There are two approaches to defining the derivative of a function on a manifold.

- The computational approach is to give it in terms of coordinates, and define how it transforms when we change coordinates. In this approach we *immediately* know how to compute with the derivative, but we have to show it is well defined.
- We can define it in a more abstract way, invariant under choice of charts. This is Do Carmo's approach and the approach we'll take. This automatically forces what the derivative has to be when we *do* express it in coordinates.

We'll first look at derivatives/tangent vectors in $\mathbb{R}^n$, and then generalize to manifolds.

3.1 Tangent vectors in $\mathbb{R}^n$

Definition 1.3: Let $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, so that $f \circ \alpha$ is a smooth function $(-\varepsilon, \varepsilon) \to \mathbb{R}$. Define the **directional derivative** or **tangent vector** in the direction of α of f to be

$$(f \circ \alpha)'(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\alpha(0))\alpha_i'(0) = \langle \nabla f, \alpha'(0) \rangle .$$

(The equality is by the chain rule.) Thus, we can think of the tangent vector as a function sending α to the linear map $f \mapsto \langle \nabla f, \alpha'(0) \rangle$.

Note the derivative depends only on $\alpha'(0)$. It didn't matter what the curve was; we could have covered all possibilities with curves that are straight lines, $\alpha(t) = p + tq$.

Definition 1.4: A **derivation** D of an $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear function from A to $\mathbb{R}$ that satisfies the Leibniz rule

$$D(fg) = (Df)g + f(Dg), \qquad f, g \in A.$$

Denote the space of derivations by $\mathrm{Der}(A)$.

Proposition 1.5: Let $C^\infty(x_1, \ldots, x_n)$ be the set of C^∞ functions on $x_1, \ldots, x_n$. The map $v \mapsto (f \mapsto \langle \nabla f, v \rangle)$ is a vector space isomorphism from $\mathbb{R}^n$ to $\mathrm{Der}(C^\infty(x_1, \ldots, x_n))$.

(Proof of surjectivity is omitted.) Think of v as $\alpha'(0)$, so the map is

$$f \mapsto \langle \nabla f, \alpha'(0) \rangle .$$

Now why did we define the directional derivative in terms of α instead of $v = \alpha'(0)$? Because we want something that *doesn't depend on coordinates*. Associated to α we get a *linear map* $f \mapsto \langle \nabla f, \alpha'(0) \rangle$ that we can define without coordinates. This is why we define the tangent vector as a linear map on a space of functions.

> We define the tangent vector to be a linear map on a space of functions, so that it does not depend on coordinates.

This will be important in the general manifold setting.

3.2 Tangent vectors in general

Our viewpoint naturally generalizes to manifolds.

Definition 1.6: Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a smooth curve. Then define the tangent vector as the linear map $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by

$$\alpha'(0)f = (f \circ \alpha)'(0)$$

The **tangent space** to M at p is

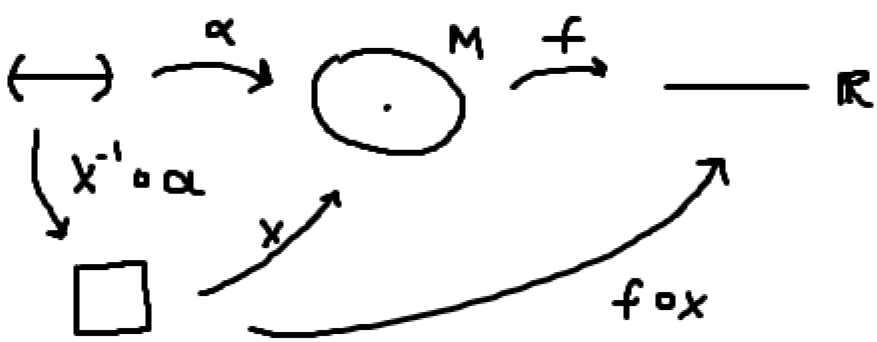
$$T_pM = \{\text{All tangent vectors to curves through } p\}$$

Note that $T_p \cong \mathbb{R}^n$. We will explain why.

To actually perform computations involving tangent vectors, we need to work on the actual charts, so the maps $(-\varepsilon, \varepsilon) \xrightarrow{\alpha} M \xrightarrow{f} \mathbb{R}$ are unsatisfactory. So given a point p, let $x : U \to M$ be a parameterization around p. Then we can work on the chart U, because we have the following commutative diagram (for small enough ε)

$$18965 - 1 - cd \quad \begin{tikzcd} (-\varepsilon,\varepsilon) \arrow[r, "\alpha"] \arrow[dr, "x^{-1}\circ\alpha"'] & M \arrow[r, "f"] & \mathbb{R} \\ & U \arrow[u, "x"] \arrow[ur, "f\circ x"'] & \end{tikzcd} \tag{1}$$

Note $(-\varepsilon, \varepsilon) \xrightarrow{x^{-1}\circ\alpha} U \xrightarrow{f\circ x} \mathbb{R}$ are maps staying in $\mathbb{R}$, so we can do multivariable calculus with them.



Write

$$x^{-1} \circ \alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$$
$$f \circ x(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \text{ as shorthand}$$
$$(f \circ \alpha)(t) = (f \circ x) \circ (x^{-1} \circ \alpha)(t).$$

The chain rule gives

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \alpha_i'(0).$$

(By abuse of notation $\frac{\partial f}{\partial x_i} = \frac{\partial f \circ x}{\partial x_i}$ and $\alpha_i'(0)$ is the ith component of $(x^{-1} \circ \alpha)'(0)$.) Think of this as the directional derivative "in coordinates," just like we can express a linear transformation as a matrix once we have coordinates.

Now if we had a different $(V, x') \neq (U, x)$ with $p \in x'(V)$, we can add V to our commutative diagram (1) We can then compute the directional derivative in the coordinates of the chart V instead of U, and we can see how the derivative changes from U to V using the chain rule on $y^{-1} \circ x$.

Thus we see that the directional derivative is an invariant notion—we don't need coordinates to define it, but once we do have coordinates, we can calculate it in terms of coordinates, and we know exactly how this expression changes when we change coordinates.

One thing to note is that if the $\alpha'(0)$ are all 0, then no matter *what* coordinates we choose, all the α_i' are still 0. But if $\alpha'(0)$ is nonzero, then we can mix things up any way we like.

Remarks:

1. $\alpha'(0)$ depends only on $\alpha'(0)$ in a coordinate system.

2. $T_p(M)$ is a n-dimensional vector space with a natural basis

$$\frac{\partial}{\partial x_i} := \text{tangent vector to curve where we only vary } x_i.$$

   (More precisely, we are considering the curve $\alpha(t) = x(x^{-1}(p) + tx_i)$.) We have $\alpha_i'(0) = \delta_{ij}$.

§4 Differentials

Definition 1.7: A smooth map $\varphi : M \to N$ induces linear maps

$$d\varphi_p : T_pM \to T_{\varphi(p)}N$$

by taking the curve α to curve $\varphi \circ \alpha$:

$$d\varphi_p(\alpha'(0)) := (\varphi \circ \alpha)'(0).$$

The map $d\varphi_p$ is called the **differential** of φ at p.

Again this does not depend on the choice of α, only on $\alpha'(0)$.

> Any smooth map φ gives rise to a differential map on the tangent spaces.

Once we have the differential, we can talk about immersions and embeddings.

Definition 1.8: We say that $\varphi : M \to N$ is a diffeomorphism if it is

1. smooth
2. bijective, and
3. φ^{-1} is smooth.

Definition 1.9: We say that $\varphi : M \to N$ is a **local diffeomorphism** at $p \in M$ if there exists an open set U containing p such that $\varphi_i : U \to \varphi(U)$ is a diffeomorphism

Definition 1.10: $\varphi : M \to N$ is an **immersion** if $d\varphi_p$ is *injective* at each p.

We have the following theorem

Theorem 1.11 (Inverse function theorem for manifolds)**:** If $d\varphi_p$ is bijective, then φ is local diffeomorphism to p.

This tells us that if the linearization of φ, i.e. $d\varphi_p$, is a bijection at p, then φ is actually a diffeomorphism at p.

Proof. Appeal to Euclidean Inverse Function Theorem (see Analysis on Manifolds, by Munkres) and compose with charts at either end. $\square$

Lecture 2
Tue. 9/11/12

Today, Bill Minicozzi is teaching again.

We define tangent bundles and vector spaces on manifolds, and then define the Lie derivative—the analogue of a derivative for vector fields. We'll derive basic properties of the Lie derivative, and understand why it is a natural thing to consider.

§1 Tangent bundle

The **tangent bundle** is basically built from considering

1. all possible points, and
2. all possible tangent vectors at that point.

At each point the tangent space is like $\mathbb{R}^n$. We have a ntural basis for the tangent basis in a given chart; namely, taking partial derivatives with respect to the coordinates of the chart. We now give the formal definition.

Definition 2.1: Let M be a n-dimensional manifold. The **tangent bundle** TM is a $(2n)$-dimensional manifold, defined as follows.

1. As a set,
$$TM = \{(p, v) : p \in M,\ v \in T_pM\}.$$

2. The charts are as follows. Start with the charts of M, $x_\alpha : U_\alpha \to M$. Define
$$y_\alpha : U_\alpha \times \mathbb{R}^n \to TM$$
by
$$y_\alpha(\underbrace{x_1^\alpha, \dots, x_n^\alpha}_{\text{in } U_\alpha}, \underbrace{u_1, \dots, u_n}_{\text{point in } \mathbb{R}^n}) = (x_\alpha(\underbrace{x_1^\alpha, \dots, x_n^\alpha}_{\text{point in } M}), \sum_{i=1}^n u_i \underbrace{\frac{\partial}{\partial x_i^\alpha}}_{\text{basis elements}})$$

This map is injective because $\frac{\partial}{\partial x_i^\alpha}$ are linearly independent basis elements. We need to check that the overlap properties work—this is standard and left to the reader.

At first glance, the tangent bundle does not seem to give much more info than M itself. It looks like we're taking the product of M with $\mathbb{R}^n$. In fact, we can contract TM to M, by contracting each set $\{p\} \times T_pM$ to just $\{p\}$.

But often $TM \neq M \times \mathbb{R}^n$. We get extra information because the tangent bundle can "spin around." When we trace out a loop in M, the tangent space might spin around completely once—so we get some nontrivial topology here.

For example, consider the tangent bundle of S^1. At each point, the tangent space is $\mathbb{R}$, and it turns out that the tangent bundle is a cylinder (exercise),
$$TS^1 = S^1 \times \mathbb{R}.$$
However, if the tangent bundle had "spun around," then we would get a Möbius band instead.

For those in the know, the tangent bundle is a special case of a *fiber bundle*. (We won't cover this more general notion, so that we can get more quickly to the geometry. But if you're interested, see `http://en.wikipedia.org/wiki/Fiber_bundle`.)

§2 Vector fields

We can now generalize the definition of a vector field to an arbitrary manifold.

Definition 2.2: A (smooth) **vector field** X on M is a map taking each point $p \in M$ to $X(p) \in T_pM$, such that the map $M \to TM$ induced by this map sending $p \mapsto (p, X(p))$ (the fiber above the point) is smooth.

We can also talk about continuous, C^1, etc. vector fields, in which case we replace the "smooth" condition by the appropriate condition.

In a chart, we can write
$$X(p) = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}.$$

The fact that X is smooth is equivalent to all the $a_i(p)$ being smooth. We get a function $X : \mathcal{D} \to \mathcal{D}$ (recall $\mathcal{D}$ is the space of smooth functions $M \to \mathbb{R}$) that operates as

$$Xf = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}$$

where implicitly at each point p we take a chart containing p.

Since X gives a tangent vector for each point of M, it allows us to take directional derivatives at each point. We call Xf the derivative of f with respect to X. Note that X is a derivation: it is $\mathbb{R}$-linear ($X(fg) = Xf + Xg$, $X(af) = aXf$) and satisfies the Leibniz rule ($X(fg) = (Xf)g + f(Xg)$).

§3 Lie derivatives

We've saw how to take the derivative of a function on the manifold with respect to a vector field. Now we would like to take the derivative of vector field *with respect to another vector field*, but we have a problem. Let X and Y be vector fields on M. Can we take a directional derivative of Y in direction X? Suppose we wanted to take

$$\lim_{t \to 0} \frac{Y(p + tX) - Y(p)}{t}.$$

This is roughly what the derivative should be. Our first problem is that this is just for Euclidean space, rather than general manifolds. For a general manifold, letting α be a curve $\alpha : (-\varepsilon, \varepsilon) \to M$, and supposing $\alpha(0) = p$, $\alpha'(0) = X(p)$, we can try to define

$$\lim_{t \to 0} \frac{Y(\alpha(t)) - Y(p)}{t}$$

However, $Y(\alpha(t))$ *and* $Y(p)$ *do not live in the same vector space*: $Y(\alpha(t))$ lives in $T_{\alpha(t)}M$ and $Y(p)$ lives in $T_p(M)$.

We are stuck unless we find a canonical way to identify these vector spaces!

There are two different ways to identify these spaces.

1. The first way is the Lie derivative, which we'll cover today. The idea is to integrate the vector field to get a diffeomorphism on the manifold, which allows us to move one point to another point. Recall that a smooth map $\varphi : M \to N$ induces a differential sending the tangent space of the first point to the tangent space of the second point, $d\varphi_p : T_pM \to T_{\varphi(p)}N$. We can go back as well, since we have a diffeomorphism.

2. The second way is to equip the manifold with extra structure, called a Riemannian connection. We use parallel transport to identify vector spaces of different points. We get what is called the covariant derivative.

3.1 Lie derivative (bracket)

Definition 2.3: Define the **Lie derivative (Lie bracket)** $[X,Y] = L_X Y$ by

$$[X,Y]f := X(Y(f)) - Y(X(f)).$$

From this definition it is not obvious that this is the "derivative" of anything; this will be clear after we derive an alternate expression for it.

Note that we're differentiating f twice, so the Lie bracket depends on at most 2 derivatives of f; it is sufficient for f to be C^2. In fact, $[X,Y]$ depends only on the first derivatives of f and is itself a vector field.

In a chart, we can write

$$X = \sum_{i=1}^n a^i \frac{\partial}{\partial x_i}$$
$$Y = \sum_{i=1}^n b^j \frac{\partial}{\partial x_j}.$$

We calculate, using the product rule,

$$X(Y(f)) = X\left(\sum_{j=1}^n b^j \frac{\partial f}{\partial x_j}\right) = \sum_{i,j} a^i \left(b^j \frac{\partial^2 f}{\partial x_j \partial x_i} + \frac{\partial b^j}{\partial x_i}\frac{\partial f}{\partial x_j}\right)$$
$$Y(X(f)) = Y\left(\sum_{i=1}^n a^i \frac{\partial f}{\partial x_j}\right) = \sum_{i,j} b^j \left(a^i \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial a^i}{\partial x_j}\frac{\partial f}{\partial x_i}\right)$$

When we subtracting, the first terms (in blue) cancel, because partial derivatives in Euclidean space commute. We switch i and j in the second terms and compute

$$\begin{aligned}[X,Y]f &= \sum_{i,j} a^i \frac{\partial b^j}{\partial x_i}\frac{\partial f}{\partial x_j} - \sum_{i,j} b^i \frac{\partial a^j}{\partial x_i}\frac{\partial f}{\partial x_j} \\ &= \sum_{j=1}^n \left[\sum_{i=1}^n \left(a^i \frac{\partial b^j}{\partial x_i} - b^i \frac{\partial a^j}{\partial x_i}\right)\frac{\partial f}{\partial x_j}\right] \\ eq:18965-2-1 [X,Y] &= \sum_{j=1}^n \left[\sum_{i=1}^n \left(a^i \frac{\partial b^j}{\partial x_i} - b^i \frac{\partial a^j}{\partial x_i}\right)\frac{\partial}{\partial x_j}\right] \qquad (2)\end{aligned}$$

In the Euclidean case, we have found a formula for $[X,Y]$. For the general manifold, we are also done, but we should say a few more words. Equation (2) gives a formula in one chart.

What if write it in another chart; do we get the same vector field? In other words, do we get the same result if we computed (2) for a different chart, and if we compute (2) in the first chart and then use the Jacobian to change coordinates? Yes, because the Lie bracket is uniquely defined by $[X,Y]f = XYf - YXf$.

If the vector field is a coordinate vector field, i.e. the a_j and b_j are constants, then the Lie bracket is 0. For a vector field coming from any coordinate system, the Lie bracket is always 0. In a sense the Lie derivative measures the obstruction to coordinates existing.

3.2 Properties of $[,]$

Proposition 2.4: Thie Lie bracket satisfies the following.

1. (Anti-commutativity) $[X, Y] = -[Y, X]$
2. ($\mathbb{R}$-linearity) $[X, aY + bZ] = a[X, Y] + b[X, Z]$.
3. (**Jacobi identity**) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (If you cyclically permute $[[X, Y], Z]$ and sum, you get 0.
4. $[fX, gY] = fg[X, Y] + fX(g)y - gY(f)X$.

As we'll explain after the proof, the Jacobi identity has a deeper reason for being true.

Proof. 1. We have $[X, Y] = X(Y(f)) - Y(X(f)) = -(Y(X(f)) - X(Y(f))) = -[Y, X]$.

2. Taking derivatives is linear.

3. Expand out each bracket, and add them all up. We have

$$\begin{aligned} [[X, Y], Z]f &= [X, Y]Z(f) - Z([X, Y]f) \\ &= XYZ(f) - YXZ(f) - ZXY(f) + ZYX(f). \\ \implies [[X, Y], Z] &= XYZ - YXZ - ZXY + ZYX \text{ (in shorthand)} \end{aligned}$$

Cyclically permuting $X \mapsto Y \mapsto Z \mapsto X$,

$$\begin{aligned} [[Y, Z], X] &= YZX - ZYX - XYZ + XZY \\ [[Z, X], Y] &= ZXY - XZY - YZX + YXZ. \end{aligned}$$

We find that all 6 permutations of X, Y, and Z occur, 6 of them are positive, and 6 of them are negative. Thus adding the 3 brackets gives 0.

4. First we find $[X, gY]$. We go nuts with the Leibniz rule, calculating

$$\begin{aligned} [X, gY](h) &= X(gY(h)) - gY(X(h)) \\ &= X(g)Y(h) + gX(Y(h)) - gY(X(h)) \\ &= X(g)Y(h) + g[X, Y]h \\ \implies [X, Y] &= X(g)Y + g[X, Y].eq:18965-2-2 \end{aligned} \tag{3}$$

We have a similar formula for $[fX, W]$.

$$[fX, W] = -[W, fX] = -W(f)X - f[W, X] = -w(f)X + f[X, W].eq:18965-2-3 \tag{4}$$

Putting (3) and (4) together gives

$$\begin{aligned}[fX, gY] &= -gY(f)X + f \underbrace{[X, gY]}_{X(g)Y+g[X,Y]} \\ &= -gY(f)X + fX(g)Y + fg[X, Y]\end{aligned}$$

□

We proved the Jacobi identity with computation; there's a more general reason why it's true.

3.3 Flows and the Lie derivative

Let X be a vector field in M. Imagine if we were to start at some point p at the manifold, and then at every instant in time, go in the direction given by the vector field. The vector field tells us how to "flow." We would trace out a curve starting at p, whose tangent vector everywhere is the vector field. We can think of this "flow" happening everywhere on the manifold, i.e. the whole manifold is "flowing." The curves that are traced out at different points will not cross.

Formally, given $p \in M$, we want some curve $\alpha_p(t)$ such that

$$\begin{aligned}\alpha_p(0) &= p \\ \alpha_p'(t) &= X(\alpha_p(t)).\end{aligned}$$

Theorem 2.5 (Existence and uniqueness of solutions)**:** thm:odes There exists a unique solution $\alpha_p(t)$ in some interval $(-\varepsilon, \varepsilon)$. Moreover, $\alpha_p(t)$ is a smooth function (with respect to (p, t)) defined on some open set in $M \times \mathbb{R}$ containing $M \times \{0\}$.

Proof. Take a coordinate chart, and appeal to the existence theorem for ODE's in Euclidean space. (See Theorems 4.3-5 in Guillemin's 18.101 notes.) □

(Note: if M is compact, then $\alpha_p(t)$ is defined for all $t \in \mathbb{R}$.)

Smooth dependence on initial conditions tells us that we can think of p as another parameter. We thus write

$$\varphi(p, t) := \alpha_p(t).$$

Proposition 2.6: We have the following properties for φ.

1. $\varphi(p, 0) = p$.
2. $\varphi(p, s + t) = \varphi(\varphi(p, s), t)$.
3. Each map $\varphi(\bullet t)$ is locally invertible. The inverse to $\varphi(\bullet, t)$ is $\varphi(\bullet, -t)$.

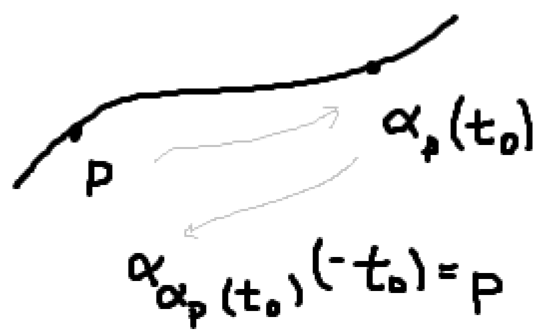
We encapsulate these conditions with the following statement.

$\varphi(\bullet, t)$ is a **1-parameter family** of (local) diffeomorphisms with $\varphi(\bullet, 0) = \text{id}$. The tangent at 0 is just the vector field.

Thus we see that $\varphi(\bullet, t)$ (thought of as a function of t, whose output is a function on the manifold) is a path in the space of diffeomorphisms. It goes through the identity at $t = 0$, and its tangent space is given by the vector field. Thus, the vector field is the Lie algebra associated to the Lie group of diffeomorphisms. The Jacobi identity holds in any Lie algebra. Thus it comes from something deeper, and is not just a miracle with 6 positive and 6 negative terms. (Don't worry if you don't understand this.)

Proof.

1. By definition.
2. This follows from uniqueness for ODE's (Theorem 2.5) and reparameterizing.
3. Follows from part 2. Think of this as "reversing time."



□

Remark: Note that solutions may blow up in finite time. For example, consider $\alpha(t)$ on $\mathbb{R}$ starting at 1, satisfying

$$\begin{aligned}\alpha(0) &= 1\\ \alpha'(t) &= X(\alpha(t)) = \alpha^p(t).\end{aligned}$$

We have

$$(\alpha^{1-p})' = (1-p)\frac{\alpha'}{\alpha^p} = (1-p)$$

Since the RHS is constant, we have $\alpha^{1-p} \to 0$ in finite time, and α blows up when $p > 1$.

There is no uniform interval $(-\varepsilon, \varepsilon)$ that works for every point; φ is not globally defined on $\mathbb{R}$ for any $p > 1$. However, if α is initially in a finite neighborhood of 0, it will be defined for some period of time $(-\varepsilon, \varepsilon)$. Given a time interval, there is a small enough neighborhood in M where φ is defined.

We can now see in what sense the Lie derivative is a "derivative," by relating it to φ.

Proposition 2.7: Let φ_t be the local flow of X. Then

$$[X, Y] = -\left.\frac{d}{dt}\right|_{t=0} d\varphi_{-t}(Y) \circ \varphi_t.$$

In other words,

$$([X, Y]f)(p) = -\left.\frac{d}{dt}\right|_{t=0} d\varphi_{-t}(Y) f(\varphi_t(p)).$$

Proof. See do Carmo, [3, Prop. 5.4, p. 28]. □

> In order to define the derivative of a vector field with respect to another vector field, we need a way to identify different tangent spaces. The Lie derivative is one such way; it identifies tangent spaces using the flow φ_t of the vector field.

We'll make 2 assumptions from now on: Manifolds are "nice" in the following sense.

1. Hausdorff: Given $p \neq q$, there exist open sets U_p, U_q such that $p \in U_p$ and $q \in U_q$ such that $U_p \cap U_q = \phi$.
2. Countable basis: M is covered by a countable collection of coordinate charts.

Lecture 3
Thu. 9/13/12

Today Toby Colding is lecturing. His office is 2-280.

Grades will be based on weekly homework and attendance. There will be 8–10 weekly homeworks. The first assignment is due on Tuesday Sept. 25, by 3pm in the undergraduate office 2-285. The grader will grade $\frac{1}{3}$ of each pset, randomly.

§1 Riemannian metric

Suppose that M^n is a smooth n-dimensional manifold. For each $p \in M$, T_pM is the vector space of tangent vectors. If V is a n-dimensional vector space, then $\langle \cdot, \cdot \rangle$ is an inner product if it is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

that is

1. linear in each variable

$$\langle v_1 + av_2, w \rangle = \langle v_1, w \rangle + a \langle v_2, w \rangle, \qquad v_1, v_2, w \in V,\ a \in \mathbb{R}$$

2. symmetric

$$\langle v, w \rangle = \langle w, v \rangle, \qquad v, w \in V$$

   and

3. positive definite

$$\langle v, v \rangle \geq 0 \text{ with equality iff } v = 0.$$

We'll denote inner products by g or by $\langle \cdot, \cdot \rangle$.

Definition 3.1: A **Riemannian metric** is a smoothly varying inner product on the tangent space, i.e. if X and Y are any two vector fields, then the function $p \xrightarrow{f} \langle X, Y \rangle (p)$ is a smooth function $f : M \to \mathbb{R}$. A manifold with a Riemannian metric is also called a **Riemannian manifold**.

Equivalently, if $p \in M$, p is in the chart $U \subseteq M$ and p is given by coordinates $x = (x_1, \ldots, x_n)$, then $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right\rangle$ is a smooth function.

To see the equivalence, note every vector field can be written as a linear combination of the $\frac{\partial}{\partial x_i}$, whose coefficients are smooth functions. For $X = \sum_i a_i \frac{\partial}{\partial x_i}$ (which we will write in shorthand as $a_i \frac{\partial}{\partial x_i}$, with the convention that we sum over the indices) and $Y = \sum_i b_i \frac{\partial}{\partial x_i}$, we have

$$\langle X, Y \rangle = a_i b_j \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = a_i b_j g_{ij}.$$

If we know the g_{ij} then we can recover the inner product.

Example 3.2: The simplest example is $M = \mathbb{R}^n$, $g = \langle \cdot, \cdot \rangle$.

Definition 3.3: Suppose we have a smooth map $f : M^m \to N^n$ that is an immersion, i.e.

$$d_p f : T_pM \hookrightarrow T_{f(p)}N$$

is injective.

Now suppose (N, g_N) has a Riemannian metric. Then there is a natural metric on M, called the **pullback**, (M, g_M), defined by

$$g_M(v, w) = g_N(d_pf(v), d_pf(w)), \qquad v, w \in T_pM.$$

If $M^m \hookrightarrow N^n$, then we call the pullback the **induced metric**.

Proof that this is a Riemannian metric. g_M sends ordered pairs of tangent vectors to $\mathbb{R}$. The differential is linear and g_N is linear, so g_M is linear. Symmetry is obvious because g_N is symmetric. g_M is clearly positive semidefinite; it is definite becase f is an immersion: if $w = v \neq 0$, then $d_pf(v) \neq 0$, so the RHS is strictly positive. $\square$

The Nash embedding theorem, proven by John Nash, says that every Riemannian manifold can be imbedded in Euclidean space, such that its metric is just the induced metric from Euclidean space.

Example 3.4: Consider $N = \mathbb{R}^3$. Then $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ is a Riemannian metric. Suppose Σ^2 is a surface and $f : \Sigma^2 \to \mathbb{R}^3$ is an immersion. Then we get an induced metric on Σ. In general, an inclusion is an immersion, so an inner product on Euclidean space gives an inner product on the submanifold.

If you take a manifold and a smooth function $h : M^m \to \mathbb{R}$, and $t \in \mathbb{R}$ is a regular value, then $h^{-1}(t)$ is a smooth manifold of dimension $m - 1$ (a hypersurface).

For example, consider $h : \mathbb{R}^3 \to \mathbb{R}$, $h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$. Then any positive value is a regular value, so $h(x_1, x_2, x_3) = c$ is 2-manifold for any $c > 0$.

Historically, Riemannian manifolds came out of looking at surfaces in Euclidean 3-space. At each point you have a natural inner product; we want to see what this inner product tells us about the geometry of the surface. Gauss pioneered this viewpoint and Riemann cast this in the more general language of manifolds. See Spivak's book; it's amusing to read about Riemann's thesis, the starting point for Riemannian geometry—a generalization of the inner product in $\mathbb{R}^3$.

We need a notion of what it means for two Riemannian metrics to be the same.

Definition 3.5: Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds. A **isometry** is a diffeomorphism $f : M \to N$ such that for all $p \in M$, and for all $v, w \in T_pM$,

$$\langle v, w\rangle_M = \langle d_pf(v), d_pf(w)\rangle_N \,.$$

The Riemannian manifolds are said to be **isometric**.

Note the fact that f is a diffeomorphism requires $m = n$.

Definition 3.6: Let G is a Lie group (a manifold that is also a group, such that group operations are smooth). For each $g \in G$, we have the left action $L_g : G \to G$ given by $L_gh = gh$ and the right action $R_gh = hg$.

Suppose $(G, \langle\cdot,\cdot\rangle)$ is a smooth n-dimensional Riemannian manifold. We say that the Riemannian metric D is **left invariant** if for all $g \in G$, $L_g : G \to G$ is an isometry. (Note it is a diffeomorphism since $L_{g^{-1}} = L_g^{-1}$.) Similarly define right invariance.

Let G be a Lie group with a left invariant Riemannian metric. It is determined completely by the inner product at the tangent space at the identity T_eG (the Lie algebra), because L_g is an isometry seding e to g. Given an inner product on the tangent space on T_eG, requiring that L_g is an isometry for each g we get a left-invariant metric. We have a correspondence between inner products on T_eG and Riemannian metrics on G.

If we have a Lie group, it's natural for us to connect the group structure with the inner product.

Definition 3.7: A metric on a Lie group that is both left and right invariant is said to be **bi-invariant**.

If G is bi-invariant, then map $G \to G$ given by $h \mapsto ghg^{-1}$ is an isometry because it is the composition of isometries $h \xrightarrow{L_g} gh \xrightarrow{R_{g^{-1}}} (gh)g^{-1}$. This gives a necessary condition for a metric to be bi-invariant.

If a Lie group is compact, then you can average over the group and construct a bi-invariant metric. Take any inner product at the tangent space of the identity, look at all the other inner products that are pullbacks, and average over the group. This construction only make sense if we can average, i.e. if M is compact.

Let M^n be a smooth manifold (that is Hausdorff with countable basis).

Claim 3.8: There exist many Riemannian metrics on M.

Let (U_α, x_α) be an atlas. Take a partition of unity $\{\phi_\alpha\}$ subordinate to this cover, i.e.

$$\phi_\alpha : M \to [0,\infty), \qquad \operatorname{Supp}\phi_\alpha \subseteq U_\alpha$$

such that given any point p, there exist at most finitely many α with $\phi_\alpha(p) \neq 0$, and $\sum_\alpha \phi_\alpha(p) = 1$.

We have $x_\alpha : U_\alpha \to \mathbb{R}^n$; we can choose any inner product on $\mathbb{R}^n$ (such as the standard inner product) to get an inner product on U_α. Then $\sum_\alpha \phi_\alpha g_\alpha$ is a inner product on M. Only finitely many terms are nonzero at each point. It is linear in each variable; it is positive semidefinite because ϕ_α are nonnegative; it is positive definite because $\sum_\alpha \phi_\alpha(p) = 1 > 0$. Make any choice of metric on the open subsets.

§2 Length of a curve

Definition 3.9: Let M be a manifold, I be an interval, and C be a curve (smooth map) $I \to M$. Note $d_t c\left(\frac{\partial}{\partial t}\right) = c'(t)$. By definition, the length of the curve is

$$\int_I \sqrt{\langle c'(t), c'(t)\rangle}\, dt = \int_I |c'(t)|.$$

Let's talk about another construction.

Definition 3.10: Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. Then define the product to be $(M_1 \times M_2, g)$ as follows. A tangent vector at (p_1, p_2) can be thought of as (v_1, v_2) where $v_i \in T_{p_i}M_i$. Taking $(v_1, v_2), (w_1, w_2) \in T_{(p_1,p_2)}(M_1 \times M_2)$, we define

$$g((v_1, v_2), (w_1, w_2)) := g_1(v_1, w_1) + g_2(v_2, w_2).$$

Linearity in each variable is clear because g_1, g_2 are linear. Symmetry follows from g_1, g_2 being symmetric. For positive definiteness, take $v_i = w_i$; the expression is nonnegative and is 0 only if v_1 and v_2 are 0.

Consider (M^n, g). Suppose $X_1, \ldots, X_n \in T_pM$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis for T_pM. Write $X_i = a_{ij}e_j$. Then the (signed) volume spanned by $X_1, \ldots, X_n$ is just $\det(a_{ij})$.

When $X_i = a_{ik}e_k$ then $\langle X_i, X_j \rangle = a_{ik}a_{jk}$. I.e. it's given by the entries of AA^T where $A = (a_{ij})$. Hence

$$\sqrt{\det(g_{ij})} = |\det(a_{ij})|\,.$$

Definition 3.11: Define the **volume** of a set U to be

$$\mathrm{Vol}(U) = \int_U \sqrt{\det(g_{ij})}.$$

We sum over pieces contained in different coordinate charts, as necessary.

A Riemannian metric gives us a way to define length and volume on a manifold.

We have more or less covered everything in chapter 1. The first several classes included lots of notation; we'll soon go on to more geometry.

Lecture 4
Tue. 9/18/12

The course website is `http://math.mit.edu/~tfei`. The first homework is due on Tuesday of next week.

Recall that to differentiate a vector field in the directino of another, we needed to identify different vector spaces; one way to do so was with the Lie derivative. Another way is to equip the manifold with extra structure, a Riemannian connection. We'll cover this today, and show how from an affine connection we can also define the covariant derivative, which is a generalization of differentiating a vector field along a curve.

§1 Affine connection

For M a smooth n-dimensional manifold, let $\mathfrak{X}(M)$ be the set of vector fields on M.

Definition 4.1: An **affine connection** on M is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ with the following properties.

1. $\nabla_Z(X + fY) = \nabla_Z X + Z(f)Y + f\nabla_Z Y$
2. $\nabla_{X+fY} Z = \nabla_X Z + f\nabla_Y Z$.

Example 4.2: ex:Rn-ac The affine connection on $\mathbb{R}^n$ is given as follows. Suppose $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}$. Then

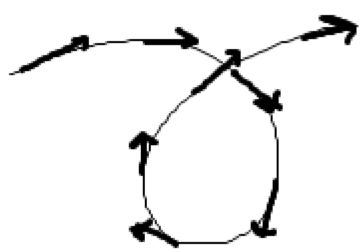
$$\nabla_X Y = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j};$$

i.e. the affine connection is just differentiating $(b_1, \ldots, b_n)$ in the direction of X.

Note an affine connection basically tells us how to differentiate one vector field with respect to another.

Definition 4.3: Suppose M^n is a smooth manifold, ∇ is a connection, and $c : I \to M$ is an interval. A **vector field** V along the curve c is a function such that for $t \in I$, $V(t) \in T_{c(t)}M$ and $t \mapsto V(t)$ is smooth.

Note $V(t)$ is not necessarily the restriction of a vector field on M, but is a vector field along the curve. For instance, the velocity field along the following self-intersecting curve is a vector field along the curve that is not the restriction of a vector field on $\mathbb{R}^2$. This is becaue at the point of intersection, there are two vectors.



Definition 4.4: df:covar-der Let $c : I \to M$ be a smooth curve and ∇ be a connection. There is a unique operation $\frac{D}{\partial t}$ called the **covariant derivative** along the curve (sending vector fields along the curve to vector fields along the curve) with the following properties.

1. (Additivity and Leibniz rule) If V, W are vector fields along the curve and $f : I \to \mathbb{R}$, then
$$\frac{D}{\partial t}(V + fW) = \frac{D}{\partial t}V + f\frac{D}{\partial t}W + f'W.$$
2. If $X \in \mathfrak{X}(M)$ then
$$\frac{D}{\partial t}X = \nabla_{c'}X.$$

Think of this as the derivative of a vector field along the curve.

Proposition 4.5: Suppose ∇ is a connection, $X, Y, Z \in \mathfrak{X}(M)$, and $p \in M$. If $X(p) = Y(p)$, then
$$(\nabla_X Z)(p) = (\nabla_Y Z)(p).$$

Consider the canonical connection on $\mathbb{R}^n$ as in Example 4.2, $\nabla_Y X = dX(Y)$. If you evaluate at p, it depends on what X is close to p, but as for Y, it only depends on the value of Y at p.

Proof. Use the fact
$$\nabla_{X+fY}Z = \nabla_X Z + f\nabla_Y Z.$$
Suppose we have local coordinates $(x_1, \dots, x_n)$ on M. Write $X = a_i\frac{\partial}{\partial x_i}$ and $Y = b_i\frac{\partial}{\partial x_i}$ (repeated indices are summed) where a_i and b_i are smooth functions on M. Now by linearity in the subscript,
$$\nabla_X Z = \nabla_{a_i\frac{\partial}{\partial x_i}}Z = a_i\nabla_{\frac{\partial}{\partial x_i}}Z,$$
and the same is true for Y. Evaluating at p, we get that they are equal at p. □

Definition 4.6: Given local coordinates on a manifold with an affine connection, define the **christoffel symbols** as the constants Γ_{ij}^k that make the following true:
$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} =: \Gamma_{ij}^k\frac{\partial}{\partial x_k}.$$

If we have two vector fields X and Y, writing $X = a_i\frac{\partial}{\partial x_i}$ and $Y = b_j\frac{\partial}{\partial x_j}$, then
$$\begin{aligned}
\nabla_X Y &= \nabla_{a_i\frac{\partial}{\partial x_i}}\left(b_j\frac{\partial}{\partial x_j}\right)\\
&= a_i\nabla_{\frac{\partial}{\partial x_i}}\left(b_j\frac{\partial}{\partial x_j}\right)\\
&= a_ib_j\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} + a_i\frac{\partial}{\partial x_i}(b_j)\frac{\partial}{\partial x_j}\\
&= a_ib_j\Gamma_{ij}^k\frac{\partial}{\partial x_k} + a_i\frac{\partial}{\partial x_i}(b_j)\frac{\partial}{\partial x_j}.
\end{aligned}$$
Note in the case of $\mathbb{R}^n$ that $\Gamma_{ij}^k = 0$ for all i, j, k.

§2 Covariant derivative

Proof of existence and uniqueness of Definition 4.4. Given an affine connection ∇, $c : I \to M$, and ∇, V, W vector fields along c, $X \in \mathfrak{X}(M)$, $\frac{D}{\partial t}$ is another vector field along c.

From the condition $\frac{D}{\partial t} X = \nabla_{c'} X$, if we want to know value at some point, than we just need to know the velocity at that point.

Given $V = a_i \frac{\partial}{\partial x_i}$, $a_i : I \to \mathbb{R}$, we must have

$$eq:965-4-1 \frac{D}{\partial t} V = a_i' \frac{\partial}{\partial x_i} + a_i \frac{D}{\partial t}\left(\frac{\partial}{\partial x_i}\right) = a_i' \frac{\partial}{\partial x_i} + a_i \nabla_{c'} \frac{\partial}{\partial x_i} \tag{5}$$

where in the last equality we used that $\frac{\partial}{\partial x_i}$ is a global vector field. We've shown that $\frac{D}{\partial t}$ only defined one way, by (5). Conversely, if $\frac{D}{\partial t}$ is defined by this, it is easy to see that it has the right properties. □

Definition 4.7: Given a manifold M, a curve $c : I \to M$, an affine connection ∇, and the corresponding covariant derivative $\frac{D}{\partial t}$, we say that a vector field V along c is **parallel** if

$$\frac{D}{\partial t} V = 0.$$

Writing $a : I \to \mathbb{R}$, $V = a_i \frac{\partial}{\partial x_i}$, $c' = c_j' \frac{\partial}{\partial x_j}$, we have

$$\begin{aligned} \frac{D}{\partial t} V &= a_i' \frac{\partial}{\partial x_i} + a_i \nabla_{c'} \frac{\partial}{\partial x_i} \\ &= a_i' \frac{\partial}{\partial x_i} + a_i c_j' \Gamma_{ji}^k \frac{\partial}{\partial x_k}. \end{aligned}$$

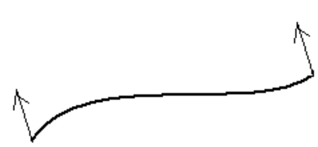
To say that V is parallel is just saying

$$\begin{aligned} & 0 = a_i' \frac{\partial}{\partial x_i} + a_i c_j' \Gamma_{ji}^k \frac{\partial}{\partial x_k} \\ \iff & 0 = (a_\ell' + a_i c_j' \Gamma_{ji}^\ell) \frac{\partial}{\partial x_k} \\ \iff & 0 = a_\ell' + a_i c_j' \Gamma_{ji}^\ell \text{ for all } \ell. \end{aligned}$$

This is an ODE! Thus, if we prescribe what the a_i are initially, then by existence and uniqueness of solutions to a first-order linear ODE, we have the following proposition.

Proposition 4.8: pr:parallel-vf Let c be a curve $I = [0,1] \to M$. For each value $V(0) \in T_{c(0)}$, there exists a unique vector field V along c that is parallel and initially has this value.

If V and W are parallel, then $V + \lambda W$ is also parallel.



Definition 4.9: Let $c : I = [0,1] \to M$ be a curve. Define the **parallel transport** map

$$P : T_{c(0)}M \to T_{c(1)}M$$

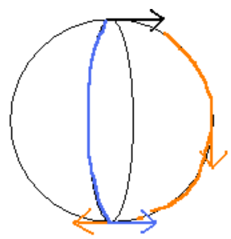
as follows. For $v \in T_{c(0)}M$, let V be a parallel vector field along c with $V(0) = v$. Define

$$P(v) = V(1).$$

P is a linear map by Proposition 4.8.

Example 4.10: Take a curve in a plane; think of the plane as a manifold. Take a vector at one end of the curve. What vector on the other end of the curve does it make sense to identify the vector with? The same one!

The vector at the other end does not depend on curve. But, in general, parallel transport depends on the curve. For instance, consider the case of a sphere.



In fact, we will see later that the difference is given by the curvature.

> An affine connection and a curve allows us to identify the tangent space at one point with the tangent space of another point.
>
> It also allows us to differentiate one vector field in the direction of another.

In general, without a given curve, we cannot identify tangent spaces at different points without imposing coordinates. If we have a curve, though, we get at least one way to identify those tangent spaces.

Suppose again that we have $M, \nabla, c, \frac{D}{\partial t}(V)$. Suppose for convenience that $c : [0,1] \to M$. Write $V = a_i \frac{\partial}{\partial x_i}$. Suppose that at $c(0)$, $X_1, \dots, X_n$ is a basis for $T_{c(0)}$. Then there exists a unique $X_i(t)$ defined as follows:

$$X_i(t) = P_{c|_{[0,t]}} X_i.$$

We get n vector fields along the curve. At each point on the curve c that $\{X_i(t)\}_i$ are linearly independent. To see this, note that they are linearly dependent initially, and that by uniqueness in Proposition 4.8, parallel translation is symmetric; going forwards and then backwards on the curve gives the identity. If the $X_i(t)$ were linearly dependent at time t, we can parallel transport backwards to get a linear dependence relation at time $t = 0$, contradiction.

We can write any vector field as $V = a_i X_i$ where a_i are smooth functions $I \to \mathbb{R}$. Now

$$\frac{D}{\partial t} V = a_i \frac{D}{\partial t} X_i + a_i' X_i = a_i' X_i$$

where the first term is 0 because X_i is a parallel basis. Thus we see that the covariant derivative is especially simple.

Recall that given M, $p \in M$, a *Riemannian metric* is a smoothly varying inner product on the tangent space.

Other structures are interesting, too: Instead of inner product, we could have indefinite (nondegenerate symmetric bilinear) forms. For example, we could consider a Lorentzian. If you study general relativity, then it's all about the Lorentzian metric. Think about a manifold as both space and time; on space we have a Riemannian metric, and on time, we have another bilinear form that is negative definite.

§3 Compatibility of metric and connection

Note that in our study of connections so far, the metric didn't play a role at all. We now bring in the metric. We want the connection to be compatible with the Riemannian metric.

Definition 4.11: Let (M^n, g) be a smooth Riemannian manifold and a connection ∇. We say that g is **compatible** with ∇ if whenever $c : I \to M$ is a curve and V and W are parallel vector fields along c, then $g(V, W)$ is constant along c.

If $C : [0, 1] \to M$ and $V(0) \perp W(0)$, then $V \perp W$ everywhere along the curve and $|V||W|$ is constant.

Example 4.12: Take the canonical example $\mathbb{R}^n$, ∇. Let $g = \langle \cdot, \cdot \rangle$ be the usual inner product. Then the connection is compatible with the metric: The inner product of parallel vector fields is constant along the curve.

Next time we will prove that any Riemannian metric gives rise to a unique connection, and show that our definition of a connection is equivalent to a more standard definition.

Lecture 5
Thu. 9/20/12

As a reminder, homework is due Tuesday by 3PM in 2-285.

Today we'll finally relate the Riemannian metric on a manifold with an affine connection. Our main theorem is the Levi-Civita Theorem 5.6, which says that a Riemannian metric automatically gives a unique symmetric compatible connection. We'll give ways to explicitly calculate what the connection is, i.e. calculate the christoffel symbols.

§1 Symmetric connections

Given a manifold M, an affine connection ∇ is a way of differentiating one vector field in the direction of another. It is linear, and satisfies the Leibniz rule in one variable (and only depends on the value at the point for the other variable).

Definition 5.1: For vector fields $X, Y \in \mathfrak{X}(M)$, ∇ is a **symmetric** connection if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

In particular, if on M we have coordinates $(x_1, \ldots, x_n)$ and $X = \frac{\partial}{\partial x_i}$ and $Y = \frac{\partial}{\partial x_j}$ then $[X, Y] = 0$, so it doesn't matter which order we take the derivative:

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}.$$

Recall that we defined the Christoffel symbols by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_i} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

If ∇ is symmetric,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} &= \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} \\ \implies \Gamma_{ij}^k \frac{\partial}{\partial x_k} &= \Gamma_{ji}^k \frac{\partial}{\partial x_k} \\ \implies \Gamma_{ij}^k &= \Gamma_{ji}^k. \end{aligned}$$

The converse is true as well.

Proposition 5.2: ∇ is symmetric if and only if there are local coordinates everywhere such that $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Let $(x_1, \ldots, x_m)$ be coordinates of a point on M. The points of TM are (p, v) with $p \in M$ and $v \in T_pM$. The coordinates of a point on TM are

$$\left(x_1, \ldots, x_n, y_1 \frac{\partial}{\partial x_1}, \ldots, y_n \frac{\partial}{\partial x_n}\right).$$

If M is a Riemannian manifold, another tangent bundle is often used.

Definition 5.3: Let (M, g) be a Riemannian manifold. Then

$$T^1M = T^1M/SM := \{(p, v) : p \in M,\, T_pM,\, g(v, v) = 1\}$$

is called the **unit tangent bundle** or **unit sphere bundle**.

Note that SM has dimension $2n-1$. In the study of dynamic systems, one looks at flows on unit tangent bundles.

From now on, we assume all connections to be symmetric. We give an alternate condition for a symmetric connection to be compatible with the metric (in some texts this is taken as a definition).

Proposition 5.4: pr:965-5-4 Let (M, g) be a manifold with a Riemannian metric and ∇ be a connection. We say that a symmetric connection is **compatible** with the metric if for $X, Y, Z \in \mathfrak{X}(M)$,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Proof. See Do Carmo [3, p. 52, Corollary 3.3]. (Note this is a corollary of Proposition 3.2 in the book, which is Proposition 5.8 here. The order is somewhat inverted in the lecture and in these notes.) □

Example 5.5: In the canonical example $M = \mathbb{R}^n$, $g = \langle \cdot, \cdot \rangle$, and the condition holds because it is just the Leibniz rule.

§2 Levi-Civita connection

Suppose again we have a smooth manifold with a connection compatible with the metric. We'll try to "isolate" $\nabla_X Y$, so we can express it only using information from the Riemannian metric (i.e. without other terms $\nabla_* \bullet$), as follows. We write (permuting the variables)

$$\begin{aligned} X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ -(\ Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\ ) \end{aligned}$$

Adding the first two equations and subtracting the third,

$$\begin{aligned} Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) \\ &\quad + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\ eq:965-5-1 &= g(Y,[X,Z]) + g(X,[Y,Z]) + g(\nabla_X Y, Z) + g(Z, \nabla_Y X). \end{aligned} \tag{6}$$

We used the fact that g is linear and symmetric. Now using $\nabla_Y X = \nabla_X Y + [Y,X] = \nabla_X Y - [X,Y]$,

$$\begin{aligned} g(\nabla_X Y, Z) + g(Z, \nabla_Y X) &= g(Z, \nabla_X Y) + g(Z, \nabla_Y X) \\ &= 2g(Z, \nabla_X Y) - g(Z, [X,Y]). \end{aligned}$$

We hence get that (6) equals

$$g(Y,[X,Z]) + g(X,[Y,Z]) - g(Z,[X,Y]) + 2g(Z, \nabla_X Y).$$

Moving the connection to the left-hand side gives

$$g(Z,\nabla_X Y)=\frac{1}{2}\left(Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g(Z,[X,Y])-g(Y,[X,Z])-g(X,[Y,Z])\right) \tag{7}$$

Note that if want to know how a connection is defined, it suffices to know the inner product of $\nabla_X Y$ with any vector field field (Just let Z vary over an orthonormal basis at a point). Thus we see from (7) that there is only one connection that is compatible. This proves that given (M,g) there exists at most one compatible connection.

Conversely, defining the connection by (7), it is easy to check that the connection is compatible with the metric.

Theorem 5.6 (Levi-Civita)**:** thm:levi-civita Given a Riemannian manifold, there is a unique symmetric and compatible connection called the **Levi-Civita connection**. It is given by (7).

Note that positive definiteness wasn't necessary here (but non-degeneracy matters).

Suppose we have a Riemannian manifold and coordinates $(x_1,\ldots,x_n)$. Defining (locally)

$$g_{ij}=g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right),$$

because g is symmetric we have that $(g_{ij})_{ij}$ is a symmetric $n\times n$ matrix at each point.

Define $(g^{ij})_{ij}=(g_{ij})^{-1}$, i.e. so that $\sum_k g_{ik}g^{kj}=\delta_{ij}$. Since (g_{ij}) is symmetric, so is (g^{ij}).

Specializing the formula (7) to $X=\frac{\partial}{\partial x_i}$, $Y=\frac{\partial}{\partial x_j}$, and $Z=\frac{\partial}{\partial x_k}$, noting the Lie brackets are 0 we get

$$g\left(\frac{\partial}{\partial x_k},\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x_i}g_{jk}+\frac{\partial}{\partial x_j}g_{ki}-\frac{\partial}{\partial x_k}g_{ij}\right) \tag{8}$$

We rewrite the LHS using Christoffel symbols:

$$g\left(\frac{\partial}{\partial x_k},\Gamma^{\ell}_{ij}\frac{\partial}{\partial x_\ell}\right)=\Gamma^{\ell}_{ij}g\left(\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_\ell}\right)=\Gamma^{\ell}_{ij}g_{k\ell}.$$

From inverse matrices, (note $g_{k\ell}=g_{\ell k}$)

$$\sum_{\ell,k}\Gamma^{\ell}_{ij}g_{\ell k}g^{ks}=\sum_{\ell}\Gamma^{\ell}_{ij}\delta_{\ell s}=\Gamma^{s}_{ij}. \tag{9}$$

To find a Christoffel symbol, we use (8) and (9) to get

$$\Gamma^s_{ij}=\sum_{\ell,k}\Gamma^{\ell}_{ij}g_{\ell k}g^{ks}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}g_{jk}+\frac{\partial}{\partial x_j}g_{ki}-\frac{\partial}{\partial x_j}g_{ki}-\frac{\partial}{\partial x_k}g_{ij}\right)g^{ks}.$$

This shows in a even more transparent way that there is only one connection; it tells us what $\Gamma^s_{ij}=\left\langle\frac{\partial}{\partial x_s},\Gamma_{ij}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right\rangle$ has to be for each i,j,s.

Example 5.7: In $\mathbb{R}^n$, for $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. We have the usual inner product

$$\langle v, w\rangle = v_1 w_1 + \cdots + v_n w_n.$$

We have $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$; think of $\mathbb{R}^n$ as space, and $\mathbb{R}$ as time. Define the inner product

$$\langle (v_1, \ldots, v_n, t_1), (w_1, \ldots, w_n, t_2)\rangle = v_1 w_1 + \cdots + v_n w_n - t_1 t_2.$$

This is a nondegenerate symmetric bilinear form. It gives us a natural metric on spacetime, which is positive definite on space but not time. General relativity is about this kind of structure on a manifold.

Let (M, g) be equipped with a Levi-Civita connection ∇. Let $I \to M$ be a curve and V a vector field along the curve. Remember that there is just one covariant derivative, determined by ∇ and the conditions it has to satisfy (Leibniz rule, etc.).

Proposition 5.8: Let V and W be vector fields along the curve. We have

$$eq: 965-5-5 \frac{d}{dt} g(V, W) = g\left(\frac{D}{\partial t} V, W\right) + g\left(V, \frac{D}{\partial t} W\right). \tag{10}$$

Proof. Writing $V = a_i \frac{\partial}{\partial x_i}$, $W = b_j \frac{\partial}{\partial x_j}$, and $a_i = a_i(t)$, we have $g(V, W) = a_i b_j g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. Taking the derivative,

$$eq: 965-5-6 \frac{d}{dt} g(V, W) = a_i' b_j g_{ij} + a_i b_j' g_{ij} + a_i b_j \frac{d}{dt} g_{ij}. \tag{11}$$

We have

$$\begin{aligned} \frac{d}{dt} g_{ij} &= c'(g_{ij}) = c' g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\ &= g\left(\nabla_{c'} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) + g\left(\frac{\partial}{\partial x_i}, \nabla_{c'} \frac{\partial}{\partial x_j}\right) \qquad \text{compatibility} \end{aligned}$$

Equation (11) then becomes

$$\frac{d}{dt} g(V, W) = a_i' b_j g_{ij} + a_i b_j' g_{ij} + a_i b_j \left((\nabla_{c'} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) + a_i b_j g\left(\frac{\partial}{\partial x_i}, \nabla_{c'} \frac{\partial}{\partial x_j}\right). \tag{12}$$

We have

$$965-5-7 \frac{D}{\partial t} V = a_i \frac{D}{\partial t} \frac{\partial}{\partial x_i} + a_i' \frac{\partial}{\partial x_i} \qquad \frac{D}{\partial t} W = b_j \frac{D}{\partial t} \frac{\partial}{\partial x_j} + b_j' \frac{\partial}{\partial x_j} \tag{13}$$

Using (13), the right-hand side of (10) becomes

$$g\left(\frac{D}{\partial t} V, W\right) + g\left(V, \frac{D}{\partial t} W\right) = a_i b_j g\left(\frac{D}{\partial t} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) + a_i' b_j g_{ij} + a_i b_j g\left(\frac{\partial}{\partial x_i}, \frac{D}{\partial t} \frac{\partial}{\partial x_j}\right) + a_i b_j' g_{ij}$$

which equals (12), as needed. $\square$

Proposition 5.9: Suppose c is a curve and V_1, V_2 are parallel vector fields along c, i.e. $\frac{D}{\partial t}V_i = 0$. Then $g(V_1, V_2)$ is constant along the curve. In particular, the length of a parallel vector field $|V| := \sqrt{g(V,V)}$ is constant along the curve.

Proof. This follows from Proposition 5.8. $\square$

Take an orthonormal basis $\{e_1, \ldots, e_n\}$ for the tangent space at $c(0)$. We know that if we parallel translate, then we get a unique parallel vector field. This gives n parallel vector fields along the curve. Now $g(e_i, e_j)$ is constant; initially it was $g(e_i, e_j) = \delta_{ij}$, so it remains so on the curve. We have constructed an orthonormal frame along the curve; for each point we get an orthonormal basis. Next time we'll use this idea to talk about geodesics.

Lecture 6
Tue. 9/25/12

Let M be a smooth manifold. We've looked at vector fields on M, $\mathfrak{X}(M)$. We've also talked about

- a connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, and
- a covariant derivative $\frac{D}{\partial t}$ along a curve.

A priori, this has nothing to do with geometric structure.

We brought in the geometric structure using a Riemannian metric g. More generally, we can let g be a symmetric nondegenerate bilinear form

$$g : T_pM \times T_pM \to \mathbb{R}$$

(smoothly varying in p).

Last time, we saw that given a smooth manifold and such a structure, we can define a compatible connection.

The canonical example is $M = \mathbb{R}^n$, $g = \langle \cdot, \cdot \rangle$, with $\langle (v_1, \ldots, v_n), (w_1, \ldots, w_n) \rangle = v_i w_i$. The Levi-Civita connection is just taking the derivative of one vector-valued function in the direction of another.

Another example of particular interest is the following.

Example 6.1 (Minkowski space)**:** The space is $\mathbb{R}^n \times \mathbb{R}^{n+1}$, with coordinates (x, t). The metric is given by

$$\langle (v_1, \ldots, v_n, s_1), (w_1, \ldots, w_n, s_2) \rangle_{\mathbb{R}^{n+1}} = v_i w_i - s_1 s_2.$$

This is a nondegenerate symmetric bilinear form but it is not positive definite.

This is the canonical example of space-time.

Keep in mind these examples.

§1 Geodesics

If we take a curve c and a vector field V along c, then we can define a covariant derivative $\frac{D}{\partial t}V$. We said that V is parallel if

$$\frac{D}{\partial t}V = 0.$$

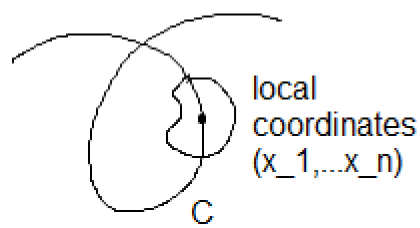
If we take vector fields V_1 and V_2, then

$$\frac{d}{dt}\langle V_1, V_2\rangle = \left\langle \frac{D}{\partial t}V_1, V_2\right\rangle + \left\langle V_1, \frac{D}{\partial t}V_2\right\rangle.$$

Definition 6.2: Let $c : I \to M$. Then c' is a vector field along the curve. We say c is a **geodesic** if c' is a parallel vector field, i.e.

$$\frac{D}{\partial t}c' = 0.$$

We now express numerically the condition for c to be a geodesic.



Write the curve as $c(t) = (x_1(t), \dots, x_n(t))$ in local coordinates. Think of $c'(t)$ as on TM, which has coordinates $\left(x_1, \dots, x_n; y_1\frac{\partial}{\partial x_1}, \dots, y_n\frac{\partial}{\partial x_n}\right)$. We have $c'(t) = (x_1(t), \dots, x_n(t), x_1'(t), \dots, x_n'(t))$, where $c' = x_j'\frac{\partial}{\partial x_j}$. Then

$$eq:965-6-1\frac{D}{\partial t}c' = \frac{D}{\partial t}\left(x_j'\frac{\partial}{\partial x_j}\right) = x_j''\frac{\partial}{\partial x_j} + x_j'\frac{D}{\partial t}\frac{\partial}{\partial x_j}. \tag{14}$$

We can express $\frac{\partial}{\partial x_j}$ using the christoffel symbols

$$eq:965-6-2\frac{D}{\partial t}\frac{\partial}{\partial x_j} = \nabla_{c'}\frac{\partial}{\partial x_j} = x_i'\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = x_i'\Gamma_{ij}^k\frac{\partial}{\partial x_k}. \tag{15}$$

Substituting (15) with (14) gives

$$\begin{aligned}\frac{D}{\partial t}c' &= x_j''\frac{\partial}{\partial x_j} + x_j'x_i'\Gamma_{ij}^k\frac{\partial}{\partial x_k}\\ &= x_j''\frac{\partial}{\partial x_j} + x_k'x_i'\Gamma_{ik}^j\frac{\partial}{\partial x_j}\\ &= (x_j'' + x_k'x_i'\Gamma_{ik}^j)\frac{\partial}{\partial x_j}.\end{aligned}$$

The condition that c is a geodesic is hence equivalent to

$$eq : 965 - 6 - 3 x_j'' + x_i' x_k' \Gamma_{ik}^j = 0 \text{ for all } j \tag{16}$$

Supposing c is a function on $I = [a, b]$, by the theory of ODE's, there is a unique solution given any choice of initial conditions

$$x_i(a), \qquad x_i'(a),$$

and we have smooth dependence on initial conditions. We have just proved the following.

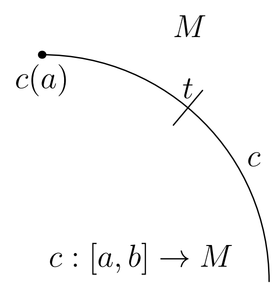
Theorem 6.3: thm:unique-geodesic Given a point $p \in M$ and a direction $v \in T_pM$, there is a unique geodesic c starting at the point and going in that direction, i.e. with $c(0) = p$ and $c'(0) = v$.

Moreover, the curve depends smoothly on the initial conditions.

Because c' is a geodesic we have

$$\frac{d}{dt} g(c', c') = g(\cancelto{0}{\frac{D}{\partial t}} c', c') + g(c', \cancelto{0}{\frac{D}{\partial t}} c') = 0$$

We give several easy observations. Let $c : [a, b] \to M$.



Let $E_1, \dots, E_n$ be a orthonormal basis at $T_{c(a)}M$. By parallel transport we get an orthonormal basis on every point of c; by abuse of notation we also denote them by $P_tE_i = E_i$. They remain orthonormal: $\frac{d}{dt} g(E_i, E_j) = \delta_{ij}$. If c is a geodesic with unit speed, i.e. $|c'| := \sqrt{g(c', c')} = 1$.

The following is elementary.

Proposition 6.4: pr:geodesic-repar Suppose $c : [a, b]$ is a geodesic and k is a constant.

1. Define the shifted curve $c_k(t) = c(t + k)$, $c_k : [a - k, b - k] \to M$.
2. Define scaled curve $c^k(t) = c(kt)$, $c : \left[\frac{a}{k}, \frac{b}{k}\right] \to M$.

Then c_k and c^k are also geodesics.

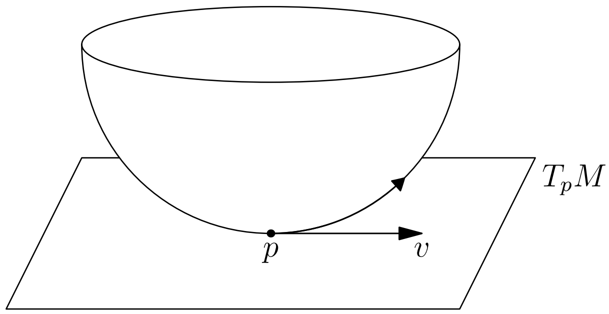
To keep our geodesic, we can't reparameterize by arbitrarily slowing down or speeding up, but we can shift the interval or go through the interval at a steady pace, but slower or faster.

§2 Exponential map

Definition 6.5: Given $p \in M$, define the **exponential map**[1] $\exp_p : T_pM \to M$ as follows. For $v \in T_pM$, let c be the geodesic with $c(0) = p$ and $c'(0) = v$

$$\exp_p(v) = c(1).$$

This depends smoothly on the vector, by Theorem 6.3 (which came from smoothness of ODE solutions in (16)).



Dropping the subscript, we think of exp as a map from the tangent bundle to M, $\exp : TM \to M$, such that if $(p, v) \in TM$ ($p \in M$, $v \in T_pM$), we have

$$\exp((p, v)) = \exp_p v.$$

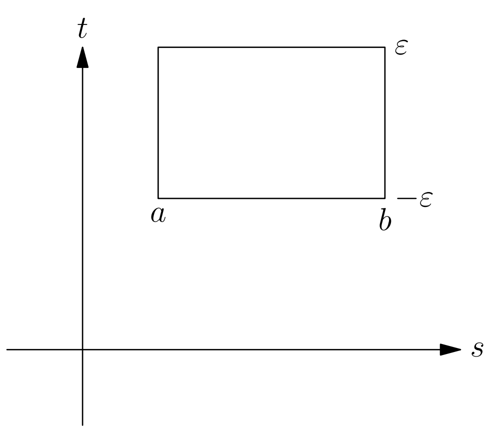
This is also smooth; we make use of the fact that the x_i depend initially on both the initial conditions on the x'_i and the x_i.

2.1 Parameterized surface

Definition 6.6: df:psurf A **parameterized surface** is a smooth map

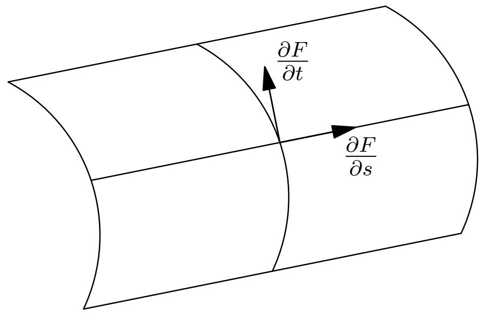
$$F : [a, b] \times (-\varepsilon, \varepsilon) \to M.$$

It may be an embedding, but this is not a requirement: it is allowed to map everything to a point.



[1]Note this is called the exponential map because for a Lie group, if we expand it out in Taylor series, it is just an exponential.

Given a parameterized surface, we can look at two vector fields $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$.



Proposition 6.7: pr:covar-commute We have the following:

$$\frac{D}{\partial t}\frac{\partial F}{\partial s} = \frac{D}{\partial s}\frac{\partial F}{\partial t}.$$

Think of this as saying that "derivatives commute."

Proof. Writing $F = (F_1, \ldots, F_n)$, we have

$$\begin{aligned}
\frac{\partial F}{\partial s} &= \frac{\partial F_i}{\partial s}\frac{\partial}{\partial x_i} \\
\frac{\partial F}{\partial t} &= \frac{\partial F_j}{\partial t}\frac{\partial}{\partial x_j} \\
\frac{D}{\partial t}\frac{\partial F}{\partial s} &= \frac{D}{\partial t}\left(\frac{\partial F_i}{\partial s}\frac{\partial}{\partial x_i}\right) \\
&= \frac{\partial^2 F_i}{\partial s \partial t}\frac{\partial}{\partial x_i} + \frac{\partial F_i}{\partial s}\frac{D}{\partial t}\frac{\partial}{\partial x_i} \\
&= \frac{\partial^2 F_i}{\partial s \partial t}\frac{\partial}{\partial x_i} + \frac{\partial F_i}{\partial s}\nabla_{\frac{\partial F_j}{\partial t}\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} \\
&= \frac{\partial^2 F_i}{\partial s \partial t}\frac{\partial}{\partial x_i} + \frac{\partial F_j}{\partial t}\frac{\partial F_i}{\partial s}\Gamma^k_{ji}\frac{\partial}{\partial x_k}.
\end{aligned}$$

This is symmetric in s and t. The proposition follows. $\square$

2.2 Gauss Lemma

We now make an observation about the exponential map. Usually we fix a point $p \in M$ and just think about $\exp_p : T_pM \to M$. We define by $\exp_p(v) = c(1)$ with $v \in T_pM$, $c(0) = p$ and $c'(0) = v$.

We can think of this another way. Recall that if we change the parameterization in a linear way, we still a geodesic (Proposition 6.4).

Now defining the geodesic c such that $c(0) = p$ and $c'(0) = tv$, and letting γ be the geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$, we have by reparameterization that $c(1) = \gamma(t)$. In other words,

$$\exp_p(tv) = \gamma(t).$$

From this we see that *the exponential map sends lines in the tangent space at p passing through 0 to geodesics passing through p on the manifold.*[2]

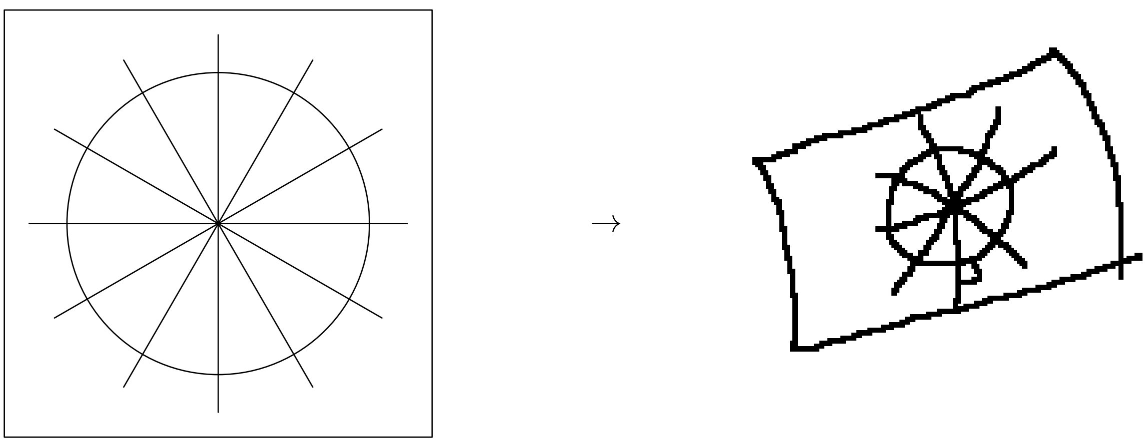
If M were a two-dimensional, then $\exp_p : T_pM \to M$ is a parameterized surface. Furthermore, we could consider polar coordinates (r, θ) on T_pM. If the manifold is not 2-dimensional, restrict exp to a 2-dimensional subspace of T_pM, $\exp : V \to M$.

In either case we have a parameterized surface.

Lemma 6.8: lem:gauss Let M be a manifold and $p \in M$. Let $\exp_p : T_pM \to M$. Let V be a 2-dimensional subspace of T_pM, expressed in polar coordinates, and consider $\exp_p : V \to M$ as a function of r and θ. Then

$$g\left(\frac{\partial}{\partial r}\exp_p, \frac{\partial}{\partial \theta}\exp_p\right) = 0.$$

In other words, the images of the circles in the tangent space are orthogonal to the geodesics starting at p.



Proof. We can think of $g\left(\frac{\partial}{\partial r}\exp_p, \frac{\partial}{\partial \theta}\exp_p\right)$ as a function of r and θ. We have, because $\exp_p(rv)$ is a geodesic for any v, that

$$\begin{aligned}
\frac{d}{dr}g\left(\frac{\partial}{\partial r}\exp_p, \frac{\partial}{\partial \theta}\exp_p\right) &= g\Big(\cancelto{0}{\frac{D}{\partial r}\frac{\partial}{\partial r}\exp_p}, \frac{\partial}{\partial \theta}\exp_p\Big) + g\left(\frac{\partial}{\partial r}\exp_p, \frac{D}{\partial r}\frac{\partial}{\partial \theta}\exp_p\right). \\
&= g\left(\frac{\partial}{\partial r}\exp_p, \frac{D}{\partial \theta}\frac{\partial}{\partial r}\exp_p\right) && \text{by Proposition 6.7} \\
&= \frac{1}{2}\frac{\partial}{\partial \theta}g\left(\frac{\partial}{\partial r}\exp_p, \frac{\partial}{\partial r}\exp_p\right) = 0
\end{aligned}$$

where the last equality follows from the fact that the velocity vector for a geodesic always has the same length. Hence $g\left(\frac{\partial}{\partial r}\exp_p, \frac{\partial}{\partial \theta}\exp_p\right)$ is constant. We just need to evaluate at origin, but it is clearly 0 at the origin. □

Next time we'll use the Gauss lemma to show geodesics are locally the shortest route.

[2] If you a line or half-line in T_pM starting at some other point besides 0, then it may not get mapped to a geodesic.

Lecture 7
Thu. 9/27/12

Get graed homeworks on Fridays in 2-285 (same box where you turn it in).

§1 Gradient and related quantities

Let (M, g) be a manifold with g a Riemannian metric.[3] (g will always be a symmetric nondegenerate bilinear form, and smoothly depend on the point. Riemannian means that it is positive definite.)

Suppose we have a function $f : M \to \mathbb{R}$ and a vector field $X \in \mathfrak{X}(M)$. We define the following quantities.

1. The **gradient** ∇f
2. The **divergence** $\operatorname{div}(X)$
3. The **Hessian** of f and **Laplacian** of f

Definition 7.1: The **gradient** ∇f is a vector field such that

$$g(\nabla f, X) = X(f).$$

(To know what the gradient is, we just need to know its projection onto every vector field.)

1.1 Gradient

Let's write the gradient in coordinates. Let $(x_1, \ldots, x_n)$ be local coordinates on M. Recall that we let $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, and (g^{ij}) is the inverse matrix of g_{ij}. Write the gradient as $\nabla f = f_i \frac{\partial}{\partial x_i}$.

By definition, $X(f) = g(\nabla f, X)$; letting $X = \frac{\partial}{\partial x_j}$ we get that

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^{n} g_{ij} f_i.$$

Then (we are implicitly summing over i, j below)

$$\frac{\partial f}{\partial x_j} g^{jk} = g_{ij} f_i g^{jk} = f_i g_{ij} g^{jk} = f_i \delta_{ik} = f_k.$$

We hence have

$$eq:787-7-1 \nabla f = f_k \frac{\partial}{\partial x_k} \text{ where } f_k = \frac{\partial f}{\partial x_j} g^{jk}. \tag{17}$$

As long as g is nondegenerate symmetric bilinear form, everything works (it doesn't have to be positive definite).

[3] We will use g and $\langle \cdot, \cdot \rangle$ interchangeably.

1.2 Divergence, Hessian, and Laplacian

Example 7.2: On $\mathbb{R}^n$, $g_{ij} = \delta_{ij}$, so $g^{jk} = \delta_{jk}$, and we are reduced to

$$\nabla f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j}.$$

This is the usual definition of the gradient on $\mathbb{R}^n$.

Definition 7.3: Define the **divergence** of a vector field $X \in \mathfrak{X}(M)$ as the (smooth) function on M given by

$$\operatorname{div}(X) = g(\nabla_{e_i} X, e_i)$$

where $e_1, \ldots, e_n$ is an orthonormal basis at T_pM and ∇ is the Levi-Civita connection corresponding to g.

Think of it as a kind of "trace." This easily seen to be independent of the orthonormal basis (by interpreting it as a kind of trace).

Definition 7.4: Define the **Hessian** of f as a function

$$\operatorname{Hess}_f : T_pM \times T_pM \to \mathbb{R}$$

given by

$$\operatorname{Hess}_f(v, w) = g(\nabla_v \nabla f, w), \qquad v, w \in T_pM.$$

We explain the notation ∇_v. Recall that if we have vector fields $X, Y \in \mathfrak{X}(M)$, and we're looking at $\nabla_X(Y)$ at the point p, the dependence on X is just on $X(p)$ (the dependence on Y is more complicated). Hence we could write

$$\nabla_X Y(p) = \nabla_{X(p)} Y$$

if we wanted. "∇_v" just means "∇_X" for any X such that X at p is v.

Proposition 7.5: Hess_f is a symmetric bilinear form.

Proof. It is clearly bilinear. To see it's symmetric, let X and Y be vector fields with $X(p) = v$ and $Y(p) = w$. Then

$$g(\nabla_v \nabla f, w) = g(\nabla_X \nabla f, Y) = Xg(\nabla f, Y) - g(\nabla f, \nabla_X Y) = XYf - g(\nabla f, \nabla_X Y).$$

Switching v and w,

$$g(\nabla_w \nabla f, v) = g(\nabla_Y \nabla f, X) = Yg(\nabla f, Y) - g(\nabla f, \nabla_Y X) = YXf - g(\nabla f, \nabla_Y X).$$

Thus for $X = \frac{\partial}{\partial x_i}$ and $Y = \frac{\partial}{\partial x_j}$ we have that the two equations are equal. Since every vector field is a linear combination of these, we get by bilinarity that

$$\operatorname{Hess}_f(v, w) = \operatorname{Hess}_f(w, v).$$

$\square$

Definition 7.6: Define the **Laplacian** of a function g by

$$\Delta f = \operatorname{div}(\nabla f).$$

The Laplacian of a function is a function because the gradient of a function is a vector field and the divergence of a vector field is a function on the manifold.

We give another expression for the Laplacian. Let $e_1, \ldots, e_n$ be an orthonormal basis of T_pM. Then

$$\Delta f = \operatorname{div}(\nabla f) = \langle \nabla_{e_i} \nabla f, e_i \rangle = \operatorname{Hess}_f(e_i, e_i).$$

Now we compute the Hessian in local coordinates. We have

$$\operatorname{Hess}_f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g\left(\nabla_{\frac{\partial}{\partial x_i}} \nabla f, \frac{\partial}{\partial x_j}\right).$$

From (17) we have $\nabla f = \frac{\partial f}{\partial x_\ell} g^{\ell k} \frac{\partial}{\partial x_k}$ so this equals

$$\begin{aligned}
g\left(\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{\partial f}{\partial x_\ell} g^{\ell k} \frac{\partial}{\partial x_k}\right), \frac{\partial}{\partial x_j}\right) &= \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k} g^{\ell k}\right) g_{kj} + \frac{\partial f}{\partial x_k} g^{\ell k} g(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}). \\
&= \frac{\partial^2 f}{\partial x_i \partial x_\ell} g^{\ell k} g_{kj} + \frac{\partial f}{\partial x_\ell} \frac{\partial g^{\ell k}}{\partial x_j} g_{kj} + \frac{\partial f}{\partial x_\ell} g^{\ell k} \Gamma^s_{ij} g_{sj}, \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} = \Gamma^s_{ik} \frac{\partial}{\partial x_s} \\
&= \underbrace{\frac{\partial^2 f}{\partial x_i \partial x_\ell} \delta_{\ell j}}_{\frac{\partial^2 f}{\partial x_i \partial x_j}} + \frac{\partial f}{\partial x_\ell} \frac{\partial g^{\ell k}}{\partial x_j} g_{kj} + \frac{\partial f}{\partial x_\ell} g^{\ell k} \Gamma^s_{ij} g_{sj}
\end{aligned}$$

Example 7.7: If $M = \mathbb{R}^n$, $g_{ij} = \delta_{ij}$, $g^{ij} = \delta_{ij}$, and $\Gamma^k_{ij} = 0$ (recall the christoffel symbols were defined $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma^k_{ij} \frac{\partial}{\partial x_k}$). Hence the second and third terms above are 0, and we just get the usual Hessian:

$$\operatorname{Hess}_f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

§2 Geodesics as locally minimizing distances

Given (M, g), recall that we defined $\exp_p : T_pM \to M$ by $\exp_p(v) = c(1)$ where c is the geodesic such that $c(0) = p$ and $c'(0) = v$. By reparameterization, we can write $\exp_p(v) = \gamma(|v|)$ where γ is the geodesic such that $\gamma(0) = p$, $\gamma'(0) = \frac{v}{|v|}$, where $|v| = \sqrt{g(v, v)}$.

The Gauss lemma says that the image of rays coming out of the origin meet the circles on the manifold orthogonally.

Theorem 7.8: Geodesics locally minimize distances.

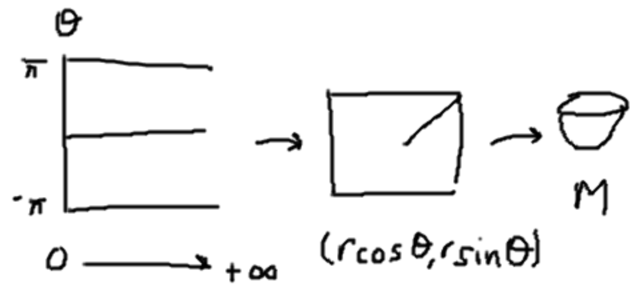
Proof. The exponential map $\exp_p : T_pM \to M$ satisfies

$$(d\exp_p)_0 = \text{id}.$$

Hence exp is a local diffeomorphism near the origin in T_pM. The exponential map gives coordinates in a neighborhood of p.

We can use polar coordinates (r, θ), which gives coordinates in a neighborhood where we remove the origin (corresponding to p). What is g_{ij} with respect to polar coordinates?

Consider the map in $(0, \infty) \times [-\pi, \pi)$ given by $(r, \theta) \mapsto (r \cos\theta, r \sin\theta)$, and then composed with the exponential map $\exp_p$.



This ia a diffeomorphism in some pointed neighborhood of the origin.

We now calculate g_{ij} for coordinates (r, θ). We have

$$\begin{aligned} 1 = g_{11} &= g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \\ 0 = g_{12} = g_{21} &= g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) \text{ by Gauss lemma} \\ g_{22} &= g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right). \end{aligned}$$

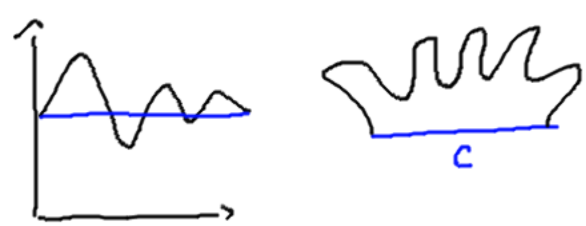
To see g_{11}, note that as we go increase r at constant speed, we're travelling along a geodesic on M with constant speed. If we travel at unit speed, we travel the geodesic with unit speed.

Note the only information really is in g_{22}.

Now we start the proof for real. Let $c : [0, 1] \to M$. We show that for some ε, $c : [0, \varepsilon] \to M$ is the shortest curve betweent he endpoints.

Suppose $c(0) = p$. Take another curve (a competing curve).

In Euclidean space, how do we know that the a straight line is the shortest path between two endpoints? If we take any competing curve, break it as a component in the direction along the straight line and another direction.



Write $\gamma = (\gamma_1, \gamma_2)$ and $\gamma' = (\gamma_1', \gamma_2')$, with $|\gamma'| \ge | \ge_1' |$, so

$$\int_a^b |\gamma'| \ge \int_a^b |\gamma_1'|$$

In a plane, the shortest past between two points is a straight line. We think of any curve as having that component and an orthogonal curve.

We break up our curve into a component in the r direction and a component in the θ direction and use the same idea. □

Lecture 8
Tue. 10/2/12

§1 Curvature

Let M^n be a Riemannian manifold with Levi-Civita connection ∇. Define the curvature as follows.

Definition 8.1: For $X, Y, Z \in \mathfrak{X}(M)$ define the **curvature** operator as follows.

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

Think of this as a difference in second order derivatives. Note there are different conventions for the curvature operator.

Proposition 8.2: pr:965-8-1 R is linear in each variable:

$$\begin{aligned} R(X_1 + X_2, Y)Z &= R(X_1, Y)Z + R(X_2, Y)Z \\ R(X, Y_1 + Y_2)Z &= R(X, Y_1)Z + R(X, Y_2)Z \\ R(X,Y)(Z_1 + Z_2) &= R(X,Y)Z_1 + R(X,Y)Z_2 \end{aligned}$$

and for $f \in C^\infty(M)$,

$$R(fX, Y)Z = R(X, fY)Z = R(X,Y)fZ = fR(X,Y).$$

Assuming that this is the case, $(R(X,Y)Z)(p)$ only depends on the value of X, Y, and Z at p. Indeed, write $X = a_i \frac{\partial}{\partial x_i}$, $Y = b_j \frac{\partial}{\partial x_j}$, and $Z = c_k \frac{\partial}{\partial x_k}$. Then using linearity in each variable,

$$\begin{aligned} R(X,Y)Z &= R\left(a_i \frac{\partial}{\partial x_i}, b_j \frac{\partial}{\partial x_j}\right)\left(c_k \frac{\partial}{\partial x_k}\right) \\ &= a_i b_j c_k R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \frac{\partial}{\partial x_k}. \end{aligned}$$

Proof. The connection is linear in each variable, so the first set of equations holds.

Now using

$$\begin{aligned}[X,Y]h &= XYh - YXh \\ [fX,Y]h &= fXY(h) - Y(f)X(h) - fYX(h) = f[X,Y](h) - Y(f)X(h)\end{aligned}$$

we calculate

$$\begin{aligned}R(fX,Y)Z &= \nabla_Y \nabla_{fX} Z - \nabla_{fX}\nabla_Y Z + \nabla_{[fX,Y]} Z \\ &= \nabla_Y f \nabla_X Z - f\nabla_X \nabla_Y Z + f\nabla_{[X,Y]} Z - Y(f)\nabla_X Z \\ &= \cancel{Y(f)\nabla_X Z} + f\nabla_Y \nabla_X Z - f\nabla_X\nabla_Y Z + f\nabla_{[X,Y]}Z - \cancel{Y(f)\nabla_X Z} \\ &= fR(X,Y)Z.\end{aligned}$$

The proof for Y is similar; we carry out the proof for Z.

$$\begin{aligned}R(X,Y)(fZ) &= \nabla_Y\nabla_X(fZ) - \nabla_X\nabla_Y(fZ) + \nabla_{[X,Y]}Z \\ &= \nabla_Y X(f)Z + \nabla_Y f\nabla_X Z - \nabla_X Y(f) Z - \nabla_X f \nabla_Y Z + [X,Y](f)Z + f\nabla_{[X,Y]}Z \\ &= \cancel{YX(f)Z} + \cancel{X(f)\nabla_Y Z} + \cancel{Y(f)\nabla_X Z} + \cancel{f\nabla_Y\nabla_X Z} \\ &\quad - \cancel{XY(f)Z} - \cancel{Y(f)\nabla_X Z} - \cancel{X(f)\nabla_Y Z} - \cancel{f\nabla_X\nabla_Y Z} + \cancel{[X,Y](f)Z} + f\nabla_{[X,Y]}Z \\ &= fR(X,Y)Z.\end{aligned}$$

□

Let's look at the special case of coordinate fields. Let $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j}$, and $Z = \frac{\partial}{\partial x_k}$. The Lie bracket is 0, so

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k}$$

If we want to define the curvature in this way on coordinate fields, then we are forced to add the term $\nabla_{[X,Y]}$ on noncoordinate fields in order for the linearity properties to hold. This ensures that R depends only on X, Y, and Z at a point.

Definition 8.3: Define the curvature symbols by

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = R_{ijk}^{\ell}\frac{\partial}{\partial x_\ell}.$$

Suppose now we have a parametrized surface $f : [a,b] \times (-\varepsilon, \varepsilon) \to M$ (see Definition 6.6) and a smooth curve $c : [a,b] \to M$. Let V is a vector field along c. We know the covariant derivative $\frac{D}{\partial t}V$ is a linear operator, satisfies the Leibniz rule, and if $V = X|_c$ then it should coincide with the connection. Recall that (Proposition 6.7)

$$\frac{D}{\partial t}\frac{\partial f}{\partial s} = \frac{D}{\partial s}\frac{\partial f}{\partial t}.$$

Just like we defined vector fields on curves, we can define vector fields on surfaces.

Definition 8.4: Define a **vector field along a parametrized surface** to be a smooth map

$$V : [a, b] \times (-\varepsilon, \varepsilon) \to TM$$

with $V(s,t) \in T_{f(s,t)}M$.

We derive a nice formula for the curvature of a vector field along a parametrized surface, in terms of covariant derivatives.

Lemma 8.5: lem:vf-ps We have

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V.$$

Proof. First assume that V is the restriction of a vector field on M. Then

$$\frac{D}{\partial s}V = \nabla_{\frac{\partial f}{\partial s}}V, \qquad \frac{D}{\partial t}V = \nabla_{\frac{\partial f}{\partial t}}V.$$

Writing $f = (f_1, \ldots, f_n)$ and letting the basis elements be $(X_1, \ldots, X_n)$ where $X_i = \frac{\partial}{\partial x_i}$, we have $\frac{\partial f}{\partial s} = \frac{\partial f_i}{\partial s}\frac{\partial}{\partial x_i}$ and hence

$$\nabla_{\frac{\partial f}{\partial s}}V = \frac{\partial f_i}{\partial s}\nabla_{\frac{\partial}{\partial x_i}}V$$

Thus we get

$$\begin{aligned}\frac{D}{\partial t}\frac{D}{\partial s}V &= \frac{D}{\partial t}\left(\frac{\partial f_i}{\partial s}\nabla_{\frac{\partial}{\partial x_i}}V\right)\\ &= \frac{\partial^2 f_i}{\partial t\partial s}\nabla_{\frac{\partial}{\partial x_i}}V + \frac{\partial f_i}{\partial s}\frac{D}{\partial t}\nabla_{\frac{\partial}{\partial x_i}}V\\ &= \frac{\partial^2 f_i}{\partial t\partial s}\nabla_{\frac{\partial}{\partial x_i}}V + \frac{\partial f_i}{\partial s}\frac{\partial f_j}{\partial t}\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}V.\end{aligned}$$

Switching the variables we get

$$\frac{D}{\partial s}\frac{D}{\partial t}V = \frac{\partial^2 f_i}{\partial s\partial t}\nabla_{\frac{\partial}{\partial x_i}}V + \frac{\partial f_i}{\partial t}\frac{\partial f_j}{\partial s}\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}V.$$

Subtracting gives (since partial derivatives commute)

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = \frac{\partial f_i}{\partial s}\frac{\partial f_j}{\partial t}\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}V - \frac{\partial f_i}{\partial t}\frac{\partial f_j}{\partial s}\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}V.$$

Note that $\left[\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right] = 0$.

In the general case, write $V = c_i(s,t)\frac{\partial}{\partial x_i}$, so we have

$$\begin{aligned}\frac{D}{\partial s}V &= \frac{\partial c_i}{\partial s}\frac{\partial}{\partial x_i} + c_i\frac{D}{\partial s}\frac{\partial}{\partial x_i}\\ \frac{D}{\partial t}\frac{D}{\partial s}V &= \frac{\partial^2 c_i}{\partial t\partial s}\frac{\partial}{\partial x_i} + \frac{\partial c_i}{\partial s}\frac{D}{\partial t}\frac{\partial}{\partial x_i} + \frac{\partial c_i}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial x_i} + c_i\frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial x_i}.\end{aligned}$$

Switching t and s, we get an equation for $\frac{D}{\partial s}\frac{D}{\partial t}V$. Subtracting the two equations we get

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = c_i \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial x_i} - c_i \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial}{\partial x_i}$$
$$= c_i R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial}{\partial x_i} \qquad \text{from the first part} = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\left(c_i \frac{\partial}{\partial x_i}\right) = R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)$$

□

Example 8.6: Consider the case $M = \mathbb{R}^n$. Then

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} + \nabla_{[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}]}\frac{\partial}{\partial x_k} = 0.$$

since $\nabla_X\left(\frac{\partial}{\partial x_i}\right) = 0$ for all i and X.

Proposition 8.7 (Bianchi identity)**:** We have

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Proof. This follows from the Jacobi identity for the Lie bracket. We'll give a detailed proof next lecture. □

§2 Sectional curvature

We want to represent the curvature with a number.

Let V be a n-dimensional vector space, and $v_1, v_2 \in V$. Then

$$|v_1 \wedge v_2| = \sqrt{|v_1|^2|v_2|^2 - \langle v_1, v_2\rangle^2}.$$

Let M be a manifold and $p \in M$. Let $r_1, r_2 \in R$. Define

$$K(p,\pi) = \frac{g(R(v_1,v_2)v_1, v_2)}{|v_1 \wedge v_2|^2}$$

where π is the linear span of v_1 and v_2.

Suppose we have a surface in 3-space, say a sphere, and we take a point. The curvature at that point is given by the formula for $K(p,\pi)$. However, this formula is difficult to work with. How can we think intuitively thing about the curvature? Imagine the points that are a distance of ε away from a point; they form a curve. We compare the length of this curve with the corresponding curve in Euclidean space. Look at the corresponding curve in Euclidean space. Look at the difference between the two lengths and dividing by some power of the radius, as $r \to 0$ this quantity goes to the curvature.

A circle on a sphere has smaller length than in Euclidean space, so a circle has positive curvature. We'll give the details in the next few lectures.

Lecture 9
Thu. 10/4/12

Given a manifold M with a symmetric connection ∇, recall that we defined $R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$. In fact, this is a function $R : T_pM \times T_pM \times T_pM \to T_pM$ since it only depends on the value at the point.

§1 Symmetries of the curvature operator

We prove the Bianchi identity.

Proof. The LHS is

$$\begin{aligned}
&\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z \\
&+ \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X \\
&+ \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[Z,X]}Y \\
= \quad &\nabla_X[Z,Y] + \nabla_Y[X,Z] + \nabla_Z[Y,X] && \nabla \text{ is symmetric} \\
&+ \nabla_{[X,Y]}Z + \nabla_{[Y,Z]}X + \nabla_{[Z,X]}Y \\
= &[X,[Z,Y]] + [Y,[X,Z]] + [Z,[Y,X]] \\
= &0.
\end{aligned}$$

We see the Bianchi identity holds because of the Jacobi identity. □

Now we explore some other symmetries of the curvature operator.

Proposition 9.1: pr:965-9-1 Let (M,g) be a manifold with a non-degenerate symmetric bilinear form, and let ∇ be the Levi-Civita connection.

Let $X,Y,Z,V \in T_pM$. Define

$$(X,Y,Z,V) := g(R(X,Y)Z,V).$$

Then the following hold.

$$\begin{aligned}
(X,Y,Z,V) + (Y,Z,X,V) + (Z,X,Y,V) &= 0 \\
(X,Y,Z,V) &= -(Y,X,Z,V) \\
(X,Y,Z,V) &= -(X,Y,V,Z) \\
(X,Y,Z,V) &= (Z,V,X,Y)
\end{aligned}$$

Proof. The first follows from the Bianchi identity.

For the second identity, we use

$$R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z = -R(Y,X)Z.$$

To see the third identity, it is enough to show $(X, Y, Z, Z) = 0$. Then by linearity

$$\begin{aligned} 0 &= (X, Y, Z + V, Z + V) \\ &= (X, Y, Z, V) + (X, Y, V, Z) + \cancelto{0}{(X, Y, Z, Z)} + \cancelto{0}{(X, Y, V, V)} \\ \implies (X, Y, Z, V) &= -(X, Y, V, Z). \end{aligned}$$

We now prove that $(X, Y, Z, Z) = 0$ by using the fact that ∇ is compatible with the connection, i.e. $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ (Proposition 5.4). We have

$$\begin{aligned} (X, Y, Z, Z) &= g(R(X, Y)Z, Z) \\ &= g(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, Z) \\ &= Yg(\nabla_X Z, Z) - g(\nabla_X Z, \nabla_Y Z) - Xg(\nabla_Y Z, Z) + g(\nabla_Y Z, \nabla_X Z) + g(\nabla_{[X,Y]} Z, Z) \\ &= \frac{1}{2} YXg(Z, Z) - \frac{1}{2} XYg(Z, Z) + g(\nabla_{[X,Y]} Z, Z) eq : 787-9-1 \qquad (18) \\ &= \frac{1}{2}[Y, X]g(Z, Z) + g(\nabla_{[X,Y]} Z, Z) \\ &= g(\nabla_{[Y,X]} Z, Z) + g(\nabla_{[X,Y]} Z, Z) = 0. \end{aligned}$$

where (18) follows from $Xg(Z, Z) = 2g(\nabla_X Z, Z)$, i.e. $g(\nabla_X Z, Z) = \frac{1}{2} Xg(Z, Z)$.

The proof of the last identity is similar. □

§2 Curvature

We now define the curvature from the curvature tensor. There are 3 types of curvatures (that we will be concerned with):

1. sectional curvature
2. Ricci curvature
3. scalar curvature

2.1 Sectional curvature

Definition 9.2: Let V be a n-dimensional vector space with an inner product. Let $X, Y \in V$. Consider the area[4]

$$|X \wedge Y| = \sqrt{|X|^2 |Y|^2 - \langle X, Y \rangle^2}$$

Define the **sectional curvature** as follows. At (M, g) with $p \in M$, let Π be a 2-dimensional subspace of $T_p M$ and define

$$K(p, \Pi) := \frac{(X, Y, X, Y)}{|X \wedge Y|^2}$$

where X, Y span Π.

[4] If you compute this thinking of X, Y in $\mathbb{R}^2$, this is just the formula for the area of a parallelogram.

A priori this depends on X and Y. We have to show this is well-defined, i.e. K depends only on Π and not on the basis $\{X, Y\}$.

Proof that K is well-defined. We show changing the basis does not change K. It suffices to show K is invariant under

1. scaling. We have
$$\frac{(X,Y,X,Y)}{|X \wedge Y|^2} = \frac{(\lambda X, Y, \lambda X, Y)}{|\lambda X \wedge Y|^2}$$
and this is likewise true if we replace Y by λY.

2. replacing $X \mapsfrom X + Y$ or $Y \mapsfrom X + Y$. We have by expanding that
$$\frac{(X+Y,Y,X,Y)}{|(X+Y) \wedge Y|^2} = \frac{(X,Y,X,Y)}{|X \wedge Y|^2} = \frac{(X,X+Y,X,X+Y)}{|X \wedge (X+Y)|^2}.$$

We can go from any basis to another using these operations, which don't change K, so K is well-defined. $\square$

Saying that a manifold has positive sectional curvature is philosophically like saying a function is convex. This is a strong condition.

2.2 Ricci curvature

In contrast, saying that a manifold has positive *Ricci* curvature is like saying a function is subharmonic.

Definition 9.3: Fix an element $X \in T_pM$. Define the bilinear form $Q(Y,Z) : T_pM \times T_pM \to \mathbb{R}$ by
$$Q(Y,Z) := (X,Y,X,Z);$$
note that Q is a symmetric bilinear form. Define the **Ricci curvature** as the trace of the bilinear form:[5]
$$\mathrm{Ric}(X,X) := \mathrm{Tr}(Q).$$

For $|X| = 1$, i.e. $g(X,X) = 1$, take an orthonormal basis $e_1 = X, \ldots, e_n$ for T_pM. Then we have
$$\begin{aligned}\mathrm{Ric}(X,X) &= \mathrm{Tr}(Q) \\ &= \sum_{i=1}^{n} Q(e_i,e_i) = (X,e_i,X,e_i) = \cancelto{0}{(X,e_1,X,e_1)} + \sum_{1<i\le n} (X,e_i,X,e_i).\end{aligned}$$

Think of the Ricci curvature as follows: Given some point and some direction, we look at the average curvature in all 2-planes that contain that direction.

[5] $\mathrm{Ric}(X,Y)$ is not defined. In my opinion this notation is a bit odd. (The book just writes $\mathrm{Ric}(X)$.)

The Ricci curvature is like an average or trace. The following analogy may be helpful: Given a function, the Hessian is a quadratic form, and the Laplacian is the trace of the Hessian. Knowing the sectional curvature is like knowing the Hessian of a function, and knowing the Ricci curvature is like knowing the Laplacian.

2.3 Scalar curvature

The scalar curvature is the most flexible notion of curvature, in the sense that conditions on the scalar curvature are weaker than conditions on the other curvatures. In fact, it is so flexible that these conditions say little unless we are in dimension 3; the Ricci curvature is usually the most useful.

To get to scalar curvature from Ricci curvature, we take another trace.

Definition 9.4: Let $p \in M$. Define the **scalar curvature** $\mathrm{Scal}_p \in \mathbb{R}$ by

$$\mathrm{Scal}_p = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i)$$

where e_i is an orthonormal basis of T_pM.

Lecture 10
Thu. 10/11/12

§1 Jacobi fields

If we have a manifold M with a symmetric connection ∇, then the curvature is defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

(It was initially defined for vector fields, but it really only depends on tangent vectors.) We proved that if $f : [a,b] \to [-\varepsilon, \varepsilon]$ is a parameterized surface (i.e. smooth map), and V is a vector field along f, then (Lemma 8.5)

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V.$$

We also showed (Proposition 6.7)

$$\frac{D}{\partial s}\frac{\partial f}{\partial t} = \frac{D}{\partial t}\frac{\partial f}{\partial s}.$$

If f is a parametrized surface, and $s \mapsto f(s,t)$ for each fixed t is a geodesic, then

$$\begin{aligned}
& 0 = \frac{D}{\partial s}\frac{\partial f}{\partial s} \\
\implies & 0 = \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial s} \\
& = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s} + \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial f}{\partial s} && \text{Lemma 8.5} \\
& = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s} + \frac{D}{\partial s}\frac{D}{\partial s}\frac{\partial f}{\partial t} && \text{Proposition 6.7} eq: 965-10-1 \qquad (19)
\end{aligned}$$

Fix t, say $t = 0$. Denote the map $s \mapsto f(s,0)$ by $\gamma(s)$; it is a geodesic by assumption. Define

$$J = \frac{\partial f}{\partial t}$$

to be the **variational vector field**. Think of $f(s,0)$ as a single curve, sitting inside a whole family of curves given by $f(s,t)$. We say $f(s,t)$ is a *variation* of curves in the t-direction. Now putting in $J = \frac{\partial f}{\partial t}$ in (19) gives

$$0 = R(\gamma', J)\gamma' + \frac{D}{\partial s}\frac{D}{\partial s}J = R(\gamma', J)\gamma' + J''.$$

Definition 10.1: If γ is a geodesic, and J is a vector field along γ, then J is said to be a **Jacobi field** if

$$J'' + R(\gamma', J)\gamma' = 0.$$

We proved that Jacobi fields naturally occur: if we take a variation of geodesics, then the variational vector field is a Jacobi field. We'll see how the Jacobi equations gives us the first explanation for a geometric notion of curvature.

Now we make some calculations. Let γ be a geodesic. Then γ' is a parallel vector field. Let $E_1, \ldots, E_{n-1}$ be orthonormal parallel vector fields along γ such that each is orthogonal to γ'.

At each point along γ, we have that γ', $E_1, \ldots, E_{n-1}$ is an orthonormal basis along $T_{\gamma(s)}M$. Suppose J is a vector field along γ. Write

$$J = j_0\gamma' + j_1E_1 + \cdots + j_{n-1}E_{n-1}$$

where the j_i are functions of s. Note $J_0 = g(\gamma', J)$, $J_i = g(J, E_i)$, $i > 0$; it is clear that the j_i are smooth functions.

Now $E_i' = 0$ and $E_i'' = 0$ so (using the fact γ is a geodesic),

$$
\begin{aligned}
J' &= j_0'\gamma' + \cancelto{0}{j_0\gamma''} + j_i'E_i + \cancelto{0}{j_iE_i'} \\
&= j_0'\gamma' + j_i'E_i \\
eq:965-10-2\, J'' &= j_0''\gamma' + j_i''E_i'.
\end{aligned}
\tag{20}
$$

Recall that the sectional curvature of a 2-plane Π was defined by

$$K(\Pi) = \frac{g(R(v_1, v_2)v_1, v_2)}{|v_1 \wedge v_2|^2}$$

where $|v_1 \wedge v_2|^2 = |v_1|^2|v_2|^2 - g(v_1, v_2)^2$. In the particular case where v_1, v_2 is an orthonormal basis, the denominator is 1 so

$$K(\Pi) = g(R(v_1, v_2)v_1, v_2).$$

Now substituting (20) into the equation for the Jacobi field $J'' + R(\gamma', J)\gamma' = 0$ we get

$$
\begin{aligned}
j_0''\gamma' + j_i''E_i + \cancelto{0}{j_0R(\gamma', \gamma')\gamma'} + j_iR(\gamma', E_i)\gamma' &= 0 \\
j_0''\gamma' + j_i''E_i + j_iR(\gamma', E_i)\gamma' &= 0
\end{aligned}
$$

Now $R(\gamma', E_i)\gamma'$ is a vector field along γ. By Proposition 9.1, this vector field is orthogonal to γ:

$$0 = g(R(\gamma', E_i)\gamma', \gamma').$$

Write $R(\gamma', E_i)\gamma' = R_i^kE_k$. Then we can write the Jacobi equation as

$$
\begin{aligned}
j_0''\gamma' + j_i''E_i + j_iR_i^kE_k &= 0 \\
j_0''\gamma' + j_i''E_i + j_kR_k^iE_i &= 0.
\end{aligned}
$$

This is true iff it is zero componentwise:

$$
\begin{aligned}
j_0'' &= 0 \\
j_i'' &= j_kR_k^i.
\end{aligned}
$$

This is a system of ordinary differential equations. The solution is unique given initial data.

We have that j_0 is a linear function, so $j_0 = ds + e$ for some constants d and e. Usually we look at Jacobi fields that are orthogonal to the geodesic. In the case where (M, g) is 2-dimensional, we can write $J = j_0\gamma' + j_1E_1$. We have $j' = j_0'\gamma' + g_1'E_1$.

Let J be a Jacobi field with $J(0) \perp \gamma$ and $J'(0) \perp \gamma'$. Let $J = j_1E_1$; we write this in short as $J = jE$. The Jacobi equation is $J'' + R(\gamma', J)\gamma' = 0$ which becomes

$$j''E + jR(\gamma', E)\gamma' = 0.$$

Now

$$R(\gamma', E)\gamma' = kE \text{ where } k = g(R(\gamma', E)\gamma', E).$$

For a surface, the sectional curvature is the Ricci curvature (under the correct normalization). We get

$$j''E + jkE = 0 \iff j'' + kj = 0.$$

This is the Jacobi equation for a 2-dimensional manifold. Consider 3 cases when k is constant.

- If $k = 0$ then $j'' = 0$ and $j = (ds + e)E$.
- When the curvature equals 1 everywhere, i.e. $k \equiv 1$, then we get $j'' + j = 0$.[6] The only solutions are $j = d\cos s + e\sin s$.

  For instance, the unit sphere has constant curvature 1. Its geodesics are the great circles are geodesics. Think of a family (variation) of great circles going through the north and south poles, with each great circle parametrized by unit speed. Then it makes sense that $j = e\sin s$ (it vanishes at $s = 0$ and π, and has minimum absolute value in the middle; we have the geodesics are together at $s = 0, \pi$ and farthest apart in the middle).
- When the sectional curvature is constantly -1, the Jacobi equation is $j'' - j = 0$. We also know what the solution is in this case; the general solution is $j = d\cosh s + e\cosh s$.

Suppose J_1 and J_2 are Jacobi fields along γ. Let

$$f(s) = g(J_1', J_2) - g(J_1, J_2').$$

Then

$$\begin{aligned} f' &= g(J_1'', J_2) - g(J_1, J_2'') \\ &\quad + \cancel{g(J_1', J_2')} - \cancel{g(J_1', J_2')} \\ &= -g(R(\gamma', J_1)\gamma', J_2) + g(J_1, R(\gamma', J_2)\gamma') = 0 \end{aligned}$$

In other words, f is constant along a geodesic. Note γ' is a Jacobi field since $\gamma'' + R(\gamma', \gamma')\gamma' = 0$.[7] Thus specializing to $J_1 = \gamma'$, this equation says $g(\gamma', J_2')$ is constant along a geodesic.

Let's revist our geometric intuition for curvature. Consider (for simplicity) the case of a 2-dimensional surface. Fix a point P. We give a geometric definition of the sectional curvature at P. Consider the image under the exponential map of a small sphere of radius ε at the origin of the tangent space.

[6] This is the 1-dimensional Schrödinger equation.

[7] Think of γ as a family of geodesics, sliding forward along itself like a snake.

Geometrically, as $\varepsilon \to 0$, the sectional curvature is first nontrivial coefficient of the Taylor expansion of the length of the the image. This is why we wanted to look at f. I.e. the sectional curvature measures the distortion of geodesics. Next time we will derive the geometric description of the curvature from our original definition.

Lecture 11
Tue. 10/16/12

sec:11

§1 Jacobi fields and curvature

sec:jac-curv Recall that for a manifold with metric (M, g) where g is symmetric and non-degenerate, we have a Levi-Civita connection ∇. Suppose we have a parametrized surface f where $s \mapsto f(s,t)$ is a geodesic. Then the variational field $J = \frac{\partial f}{\partial t}$ satisfies the Jacobi equation

$$J'' + R(\gamma', J)\gamma' = 0$$

where $J'' = \frac{D}{\partial s}\frac{D}{\partial s}J$.

For instance, consider the exponential map $\exp_p : T_pM \to M$. We can consider it in polar coordinates, so it takes (r, θ) as arguments, where $-\pi < \theta \le \pi$ and $0 < r$. Then $\frac{\partial}{\partial \theta}\exp_p = J$ satisfies the Jacobi equation. The Jacobi field measures the "distortion" of the exponential map; moreover $\left|\frac{\partial}{\partial r}\exp_p\right| = 1$.

Consider the special case where M^2 is a surface. Letting γ be a geodesic starting at p, $\gamma(t) = \exp_p(t, \theta)$, we have by Gauss's Lemma 6.8,

$$eq : 965 - 11 - 1\frac{\partial}{\partial \theta}\exp_p \perp \gamma'. \tag{21}$$

Note $\gamma'' = 0$ since γ' is a parallel vector field along γ. The normal $\vec{n}$ to the curve is also parallel along γ (by parallel translation). By (21), J is perpendicular to γ so we can write $J = j\vec{n}$,$J' = j'\vec{n}$, and $J'' = j''\vec{n}$. The Jacobi equation tells us

$$j''\vec{n} + jR(\gamma', \vec{n})\gamma' = 0.$$

Writing $R(\gamma', \vec{n})\gamma' = k\vec{n}$, we get

$$j''\vec{n} + jk\vec{n} = 0 \iff j'' + kj = 0.$$

Now $J(0) = \frac{\partial}{\partial\theta}\exp_p(0,0) = 0$ so $j(0) = 0$, and we get $\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}\exp_p\Big|_{r=0} = \vec{n}$ gives $j'(0) = 1$. Taylor expansion gives

$$j = j(0) + j'(0)r + \frac{j''(0)}{2}r^2 + \frac{j'''(0)}{6}r^3 + \cdots$$

where

$$\begin{aligned} j(0) &= 0 \\ j'(0) &= 1 \\ j''(0) &= -j(0)k(0) = 0 \\ j'''(0) &= -j'(0)k(0) - j(0)k'(0) \\ &= -k(0). \end{aligned}$$

Thus

$$eq:965-11-2 j = r - \frac{k}{6}r^3 + (\text{higher order terms})\,. \tag{22}$$

This gives us a way of thinking about the curvature. Suppose we have a surface, and we want to know the curvature at p. Consider a sphere (circle) of radius r at 0 in T_pM; call it S_r. Then $\exp_p(S_r)$ is a curve on the manifold; call it ∂B_r. What is the length of ∂B_r?

We'd like to compute the length $c(\theta) = \exp_p(r,\theta)$. We have

$$\frac{\partial c}{\partial\theta} = \frac{\partial}{\partial\theta}\exp_p = J.$$

Hence the length is

$$eq:965-11-3 \left|\frac{\partial c}{\partial\theta}\right| \approx r - \frac{k}{6}r^3. \tag{23}$$

Integrating this from $-\pi$ to π, the length of ∂B_r is

$$\int_{-\pi}^{\pi}\left|\frac{\partial c}{\partial\theta}\right| d\theta \approx \left(r - \frac{k}{6}r^3\right)2\pi = 2\pi r - \frac{k}{3}\pi r^3.$$

Thus we see

$$\operatorname{length}(\partial B_r) - \operatorname{length}(S_r) \approx -\frac{k}{3}\pi r^3.$$

For instance, for the sphere, this difference is negative so the curvature is positive.

The curvature measures the difference between the length of distance spheres in Euclidean space, and the length of the distance spheres on the manifold.

- Positive curvatures means that spheres on the manifold have smaller length. Geodesics coming from a point don't spread out as fast.
- Negative curvature means the opposite.

We can get a similar expression in any dimension.

§2 Conjugate points

Suppose γ is a geodesic, $\gamma : [a, b] \to M$. Let J be a Jacobi field. Given $(v, w) \in T_pM \times T_pM$ where $\gamma(a) = p$, there exists a unique Jacobi field with

$$J(a) = v, \qquad J'(a) = w.$$

In particular, letting J be the Jacobi field with $J(a) = 0$ and $J'(a) = v$, we get a lnear map

$$F : T_{\gamma(a)}M \to T_{\gamma(b)}M$$

sending $w \in T_{\gamma(a)}M$ to $J(b)$.

Definition 11.1: We say that b is a **conjugate point** to a along γ if there is a non-trivial Jacobi field with $J(a) = 0$ and $J(b) = 0$.

The manifold is nicer if we have no conjugate point.

§3 Isometric immersions

Consider a submanifold $M^2 \subseteq \mathbb{R}^3$ or more generally, any isometric immersion $M^m \hookrightarrow N^n$.

The following are natural questions:

- How do the connection on M and N relate?
- How do the curvatures of M and N relate?

Let $\overline{\nabla}$ be the connection on N and ∇ be the connection on M. The following says that if we want to calculate $(\nabla_X Y)_p$, we extend X and Y in any way to N, calculate the covariant derivative and then take the tangential component.

Proposition 11.2: Let $X, Y \in \mathfrak{X}(M)$, and $\overline{X}$, $\overline{Y}$ are (local) extensions of X and Y to a vector field on N, then

$$\nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T$$

where T means tangential component.

To see that we can extend the vector field, note that because M is a submanifold of N, by the implicit function theorem there is a coordinate system $(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$ on which the manifold sits like a plane, $(x_1, \dots, x_m, 0, \dots, 0)$. Extend the vector field trivially in the other directions.

Proof. Define $\nabla_X Y$ by $(\overline{\nabla}_{\overline{X}} \overline{Y})^T$. We need to check ∇ is a symmetric compatible connection. We check

1. ∇ is linear in each variable (clear).
2. If we multiply X by a function it pops out linearly (clear).

3. If we multiply Y be a function, the Leibniz rule holds. Extending f to a function on a neighborhood of M near a point,
$$\begin{aligned}\nabla_X(fY) = (\overline{\nabla}_{\overline{X}}(\overline{fY}))^T &= \left(\overline{X}(\overline{f})\overline{Y} + \overline{f}\overline{\nabla}_{\overline{X}}\overline{Y}\right)^T \\ &= X(f)Y + f(\overline{\nabla}_{\overline{X}}\overline{Y})^T.\end{aligned}$$

4. Symmetry: We have
$$\begin{aligned}\nabla_X Y - \nabla_Y X &= (\overline{\nabla}_{\overline{X}}\overline{Y})^T - (\overline{\nabla}_{\overline{Y}}\overline{X})^T \\ &= (\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X})^T \\ &= \left([\overline{X}, \overline{Y}]\right)^T = [X, Y].\end{aligned}$$

5. Compatibility with connection: Given $X, Y, Z \in \mathfrak{X}(M)$, we need to check
$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$
But taking the derivative in direction Z is the same whether we are thinking in M or in N. Thus this is equivalent to
$$\overline{Z}g(\overline{X}, \overline{Y}) = g(\overline{\nabla}_{\overline{Z}}\overline{X}, \overline{Y}) + g(\overline{X}, \overline{\nabla}_{\overline{Z}}\overline{Y}),$$
which holds.

6. Well-definedness: ∇ doesn't depend on the extension.

□

§4 Second fundamental form

Given $M \hookrightarrow N$, $X, Y \in \mathfrak{X}(M)$ and $\overline{X}, \overline{Y} \in \mathfrak{X}(M)$ such that $\overline{X}|_M = X$, $\overline{Y}|_M = Y$, we claim
$$(\overline{\nabla}_{\overline{X}}\overline{Y})^{\perp}(p)$$
depends only on the value of $\overline{X}(p)$ and $\overline{Y}(p)$.

We have, for a function f extended in a neighborhood in N,
$$(\overline{\nabla}_{\overline{X}}(\overline{fY}))^{\perp} = (\overline{f}\overline{\nabla}_{\overline{X}}\overline{Y})^{\perp} + \left(\overline{X}(f)\overline{Y}\right)^{\perp} = \overline{f}\left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)^{\perp} + \underbrace{\overline{X}(f)(\overline{Y}(p))^{\perp}}_{0}$$
because $\overline{Y}(p) = Y(p)$ has no perpendicular component. Thus $\overline{\nabla}^{\perp}$ is linear in both its bottom and top variable; we've seen that anything with such a linearity property (where functions just pop out) just depend on values at p. Thus $(\overline{\nabla}_{\overline{X}}\overline{Y})^{\perp}(p)$ depends only on X, Y at p.

Definition 11.3: The bilinear map $B : T_pM \times T_pM \to \mathbb{R}$ is defined by

$$B(X,Y) := (\overline{\nabla}_{\overline{X}}\overline{Y})^{\perp}(p).$$

and is called the **second fundamental form**.

We claim that B is also symmetric. We have

$$B(Y,X) = (\overline{\nabla}_{\overline{Y}}\overline{X})^{\perp} = (\overline{\nabla}_{\overline{X}}\overline{Y})^{\perp} + ([\overline{X},\overline{Y}])^{\perp} = B(X,Y)$$

because the Lie bracket of two vectors in the tangent space of M is still in the tangent space, and hence has orthogonal component 0.

Lecture 12
Thu. 10/18/12

Last time we considered isometric immersions $M^m \hookrightarrow N^n$. This means that the map is an immersion and the metric is just induced by inclusion. Let $X, Y, Z \in \mathfrak{X}(M)$. We proved that if $\overline{X}$ and $\overline{Y}$ are extensions of X and Y to a tubular neighborhood of p, then

$$(\overline{\nabla}_{\overline{X}}\overline{Y})^{T} = \nabla_X Y.$$

Last time we also defined the second fundamental form $B(X,Y)$ by

$$B(X,Y) = (\overline{\nabla}_{\overline{X}}\overline{Y})^{\perp}.$$

We have that B is symmetric bilinear form

$$B : T_pM \times T_pM \to (T_pM)^{\perp} \subseteq T_pN.$$

Suppose E is a vector field perpendicular to M. Then

$$\begin{aligned}
\langle B(X,Y), E\rangle(p) &= \left\langle (\overline{\nabla}_{\overline{X}}\overline{Y})^{\perp}, E\right\rangle \\
&= \left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, E\right\rangle && E \text{ is perpendicular} \\
&= \underbrace{\overline{X}\left\langle \overline{Y}, E\right\rangle}_{0} - \left\langle \overline{Y}, \overline{\nabla}_{\overline{X}}E\right\rangle \; eq:965-12-1 && (24) \\
&= -\left\langle \overline{Y}, \overline{\nabla}_{\overline{X}}E\right\rangle
\end{aligned}$$

In (24) we noted that $\overline{Y}$ is tangent to M and E is normal to M, so the first term is 0.

Since B is symmetric, if we switch X, Y we get the same thing:

$$\langle Y, \nabla_X E\rangle = \langle X, \nabla_Y E\rangle.$$

§1 Weingarten map

Suppose that $M^{n-1} \subseteq N^n$ is a hypersurface. Then up to sign, locally there is a unique normal vector field n to M.

Definition 12.1: The **Weingarten map** is the map $W : T_pM \to T_pM$ defined by

$$W(v) = \nabla_v n.$$

It is clear that W is linear. We need to show that $W(T_pM) \subseteq T_pM$ is actally in T_pM, or equivalently, $\langle W(v), n\rangle = 0$ for all $v \in T_pM$. Let X extend v. We have

$$\langle W(v), n\rangle = \langle \nabla_X n, n\rangle = \frac{1}{2} X \underbrace{\langle n, n\rangle}_{1} = 0$$

Next we show that W is symmetric (self-adjoint):

$$\langle W(v), u\rangle = \langle v, W(u)\rangle .$$

We calculate

$$\begin{aligned}
\langle W(v), u\rangle &= \langle \nabla_v n, u\rangle \\
&= - \langle B(v, u), n\rangle \\
&= - \langle B(u, v), n\rangle \\
&= \langle \nabla_u n, v\rangle .
\end{aligned}$$

This implies by the spectral theorem that W has a basis of eigenvectors.

Definition 12.2: The eigenvectors of W are called **principal directions** and the eigenvalues are called **principal curvatures**.

Note it is easy to generalize the theory to arbitrary isometric immersions $M \hookrightarrow N$; see the book.

We now relate the curvature of M to the curvature of N using the Gauss equations.

§2 Gauss equations

Let E_1, E_2, and E_3 be vector fields tangent to M. Then[8]

$$\begin{aligned}
\overline{R}(E_1, E_2)E_3 &= \overline{\nabla}_{E_2}\overline{\nabla}_{E_1}E_3 - \overline{\nabla}_{E_1}\overline{\nabla}_{E_2}E_3 + \overline{\nabla}_{[E_1,E_2]}E_3, \\
R(E_1, E_2)E_3 &= \nabla_{E_2}\nabla_{E_1}E_3 - \nabla_{E_1}\nabla_{E_2}E_3 + \nabla_{[E_1,E_2]}E_3.
\end{aligned}$$

[8]We implicitly extend the vector fields to N. We may as well just work with vector fields on M, though, because the definition doesn't depend on the extension.

Because we are working with a hypersurface, we can write

$$\begin{aligned}\overline{\nabla}_{E_1} E_3 &= \nabla_{E_1} E_3 + \langle B(E_1, E_3), n\rangle\, n,\\ \overline{\nabla}_{E_2} E_3 &= \nabla_{E_2} E_3 + \langle B(E_2, E_3), n\rangle\, n.\end{aligned}$$

Then we have using the above and the Leibniz rule,

$$\begin{aligned}\overline{\nabla}_{E_2}\overline{\nabla}_{E_1} E_3 &= \overline{\nabla}_{E_2}\nabla_{E_1} E_3 + \overline{\nabla}_{E_2}(\langle B(E_1, E_3), n\rangle\, n)\\ &= \overline{\nabla}_{E_2}\nabla_{E_1} E_3 + E_2(\langle B(E_1, E_3), n\rangle) n + \langle B(E_1, E_3), n\rangle\, \overline{\nabla}_{E_2} n\\ &= \nabla_{E_2}\nabla_{E_1} E_3 + \langle B(\nabla_{E_1} E_3, E_2), n\rangle\, n + E_2(\langle B(E_1, E_3), n\rangle) n + \langle B(E_1, E_3), n\rangle\, \overline{\nabla}_{E_2} n.\end{aligned}$$

Then (noting the normal terms don't contribute to the inner product),

$$\begin{aligned}\left\langle \overline{\nabla}_{E_2}\overline{\nabla}_{E_1} E_1, E_2\right\rangle &= \langle \nabla_{E_2}\nabla_{E_1} E_1, E_2\rangle + \langle B(E_1, E_1), n\rangle \left\langle \overline{\nabla}_{\overline{E_2}} n, E_2\right\rangle\\ &= \langle \nabla_{E_2}\nabla_{E_1} E_1, E_2\rangle - \langle B(E_1, E_1), n\rangle \langle B(E_2, E_2), n\rangle\, eq:965-12-2 \qquad (25)\end{aligned}$$

We similarly have

$$eq:965-12-3 \left\langle \overline{\nabla}_{E_1}\overline{\nabla}_{E_2} E_1, E_2\right\rangle = \langle \nabla_{E_1}\nabla_{E_2} E_1, E_2\rangle - \langle B(E_2, E_1), n\rangle^2. \qquad (26)$$

Finally,

$$eq:965-12-4 \left\langle \overline{\nabla}_{[E_1,E_2]} E_1, E_2\right\rangle = \left\langle \nabla_{[E_1,E_2]} E_1, E_2\right\rangle \qquad (27)$$

From (25), (26), and (27) we get

$$\left\langle \overline{R}(E_1, E_2)E_1, E_2\right\rangle = \langle R(E_1, E_2)E_1, E_2\rangle - \langle B(E_1, E_1), n\rangle \langle B(E_2, E_2), n\rangle + \langle B(E_1, E_2), n\rangle^2$$

Choosing E_1, E_2 to be orthonormal, we get the **Gauss equation**

$$eq:965-gauss-eq\, \overline{K} = K - \langle B(E_1, E_1), n\rangle \langle B(E_2, E_2), n\rangle + \langle B(E_1, E_2), n\rangle^2; \qquad (28)$$

the curvature with respect to M and N differ by a correction term. Using $\langle B(E_1, E_2), n\rangle = -\langle \nabla_{E_1} n, E_2\rangle$, we can write the Gauss equations in terms of the Weingarten map:

$$\overline{K} = K - \langle \nabla_{E_1} n, E_1\rangle \langle \nabla_{E_2} n, E_2\rangle + \langle \nabla_{E_1} n, E_2\rangle^2.$$

We can write this in terms of the eigenvalues of the Weingarten map, the principal curvatures. Consider the case $M^2 \subset N^3$. Let $p \in M$. We compute the sectional curvature of T_pM. Take an orthonomal basis of eigenvectors (principal directions) E_1, E_2; suppose $\nabla_{E_1} n = \kappa_1 E_1$ and $\nabla_{E_2} n = \kappa_2 E_2$. Then we get

$$\begin{aligned}\overline{K} &= K - \langle \kappa_1 E_1, E_1\rangle \langle \kappa_2 E_2, E_2\rangle + \langle \kappa_1 E_1, E_2\rangle^2\\ \overline{K} &= K - \kappa_1\kappa_2.\end{aligned}$$

Example 12.3: We calculate the curvature of the unit sphere $S^2 \subset \mathbb{R}^3$. We know $K_{\mathbb{R}^3} \equiv 0$. The curvature of S^2 is $K_{\mathbb{R}^3}$ minus the product of the two principal cuvatures:

$$0 = K_{\mathbb{R}^3} = K_{S^2} + \kappa_1\kappa_2.$$

The unit normal is simply x. We have

$$\nabla_v x = v;$$

all directions are principal directions (the Weingarten map is the identity) and the principal cuvatures are 1. Hence $\kappa_1 = \kappa_2 = 1$ and we get

$$K_{S^2} = -1.$$

Use the Gauss equation $\overline{K} = K - \kappa_1\kappa_2$ to calculate the curvature of a submanifold.

Example 12.4: We calculate the curvature of a round cylinder $S^1 \times \mathbb{R}$ in $\mathbb{R}^3$. Let n be the normal. Note that along the height of the cylinder, n is constant. Hence

$$\nabla_v n = 0.$$

The height direction is a principal direction with principal value 0. It doesn't matter what the other principal value is; the product is 0. We get

$$K_{S^1\times\mathbb{R}} = 0.$$

In fact his works for any cylinder $\gamma \times \mathbb{R}$ where γ is a closed curve in the plane.

Lecture 13
Tue. 10/23/12

Suppose we have an isometric immersion $M^m \hookrightarrow N^n$. Recall that letting ∇ and $\overline{\nabla}$ be the connections on M and N, we have $\nabla = (\overline{\nabla})^T$. The operator $(\overline{\nabla})^\perp$ is called the second fundamental form:

$$B(X,Y) = (\overline{\nabla}_X Y)^\perp, \qquad X, Y \in T_pM$$

This is bilinear and symmetric.

Now given a hypersurface $M^{n-1} \subseteq N^n$, if n is a unit normal vector field, define the Weingarten map $W : T_pM \to T_pM$ by

$$W(X) = \nabla_X n.$$

This is a symmetric linear map, and we have

$$-\langle \nabla_X n, Y\rangle = \langle B(X,Y), n\rangle .$$

The Gauss equation for $M^2 \subseteq N^3$ is

$$K_N = K_M - \kappa_1\kappa_2$$

where κ_1, κ_2 are the principal curvatures.

§1 Mean curvature

Definition 13.1: Define the **mean curvature** as the trace of the Weingarten map:[9]

$$H = \operatorname{Tr}(\nabla_\bullet n).$$

Definition 13.2: Let $M \subseteq N$, and let X be a vector field on M. Define the divergence by

$$\operatorname{div}_M(X) = \langle \nabla_{e_i} X, e_i\rangle .$$

Letting $e_1, \ldots, e_n$ be an orthonormal basis for T_pM, we have

$$H = \operatorname{Tr}(\nabla_\bullet n) = \sum_{i=1}^n \left\langle \overline{\nabla}_{e_i} n, e_i \right\rangle = \operatorname{div}_M(n).$$

Example 13.3: Last time we saw that if $S^2 \subseteq \mathbb{R}^3$ is the unit sphere, then $n = x$ and the curvature is 1. Now

$$\operatorname{div}_{S^2}(x) = H_{S^2} = 2.$$

Example 13.4: Consider $S^n \subseteq \mathbb{R}^{n+1}$. By the same argument, the mean curvature is

$$H_{S^n} = n.$$

Consider the second fundamental form of $S^2 \subseteq \mathbb{R}^3$. We have

$$\langle B(X,Y), n\rangle = -\langle \nabla_X n, Y\rangle = -\langle X, Y\rangle .$$

Definition 13.5: Consider a closed hypersurface M, i.e., $M = \partial\Omega$ where Ω is compact. We say M is convex if letting n be the inward normal, we have $\langle B(X,Y), n\rangle$ is positive semidefinite, i.e.,

$$\langle B(X,X), u\rangle \geq 0$$

for all $x \in T_pM$.

This is the usual notion of convexity.

[9] Note that in Do Carmo, H is defined a bit differently, as $H = \frac{1}{n}\operatorname{Tr}(\nabla_\bullet n)$ where n is the dimension.

§2 Gauss map

Let $M^n \subseteq \mathbb{R}^{n+1}$ is a closed orientable hypersurface. Define the Gauss map

$$n : M^n \to S^n$$

by taking the unit normal on M^n.

If is easy to show that if $M^n \subseteq \mathbb{R}^{n+1}$ is closed orientable and strictly convex, then the Gauss map $n : M \to S^n$ is a diffeomorphism. Indeed, for $X \neq 0$,

$$0 < \langle B(X,X), n\rangle = -\left\langle \overline{\nabla}_X n, X\right\rangle = -\langle dn(X), X\rangle .$$

(Note that the connection on $\mathbb{R}^{n+1}$ is given by $\overline{\nabla}_X n = dn(X)$.) This is locally a diffeomorphism; it's actually one-to-one.

We claim that our notion of convexity is the same as the typical notion: a set is convex if the segment between any two points in the set is entirely contained in the set. This is left as an exercise. The idea is that M were like in that in the figure, then M will not be convex at the marked point. It looks like the outward normal on the unit sphere, so is not convex there.

Convexity gives $\langle B(X,X), n\rangle \geq 0$ for $X \neq 0$, n the inward normal. This is $-\langle \nabla_X n, X\rangle$. The mean convexity is defined as

$$-\operatorname{div}_M n = -H \geq 0$$

where n is the inward normal. Convex implies that the mean convexity is nonnegative.

The Weingarten map $\nabla_\bullet : T_pM \to T_pM$ is a symmetric linear map, so has an orthogonal basis consisting of eigenvectors $e_1, \ldots, e_{n-1}$, called the principal directions, with eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$. We have

$$H = \operatorname{div}_M(n) = \left\langle \overline{\nabla}_{e_i} n, e_i\right\rangle = \langle \kappa_i e_i, e_i\rangle = \sum_{i=1}^{n-1} \kappa_i$$

Definition 13.6: If $M^{n-1} \hookrightarrow N^n$ is a hypersurface, we say that M is **minimal** hypersurface if $H = 0$.

Suppose $M^n \subset N^{n+1}$. If M is minimal, and X is a vector field on M, we have

$$eq:965-13-1\, \operatorname{div}_M(X) = \operatorname{div}_M(X^T). \tag{29}$$

To see this, write

$$X = X^T + \langle X, n\rangle\, n.$$

We have

$$\operatorname{div}_M(X) = \dim_M(X^T) + \operatorname{div}_M(\langle X, n\rangle\, n) = \operatorname{div}_M(X^T) + \cancelto{0}{\langle \nabla_{e_i}(\langle X, u\rangle\, n), e_i\rangle}.$$

Indeed, (using summation notation)

$$\nabla_{e_i} \langle X, n\rangle\, u = e_i(\langle x, n\rangle)n + \langle X, n\rangle\, \nabla_{e_i} n, \qquad \langle \nabla_{e_i}(\langle x, n\rangle\, n), e_i\rangle = \langle X, n\rangle\, H = 0.$$

Let M be a minimal hypersurface, $H = 0$, let x_i be the coordinates in $\mathbb{R}^n$ of a point on M.

$$\Delta_M x_i = \operatorname{div}_M(\nabla^M x_i) \overset{29}{=} \operatorname{div}_M(\nabla^{\mathbb{R}^{n+1}} x_i) = \operatorname{div}_M(e_i) = 0.$$

We claim there is no closed minimal hypersurface in $\mathbb{R}^{n+1}$. To prove this, note if $\Delta_M X_i$ then X is constant. We can argue using the maximum principle.

Lecture 14
Thu. 10/25/12

Absent.

We covered the Hopf-Rinow Theorem. See [3, p. 144-149].

Definition 14.1: A Riemannian manifold M is **extendible** if there exists a Riemannian manifold M' such that M is isometric to a proper open subset of M'.

A Riemannian manifold M is (geodesically) **complete** if for all $p \in M$, $\exp_p$ is defined for all $v \in T_pM$, i.e., any geodesic $\gamma(t)$ starting at p is defined for all $t \in \mathbb{R}$.

Proposition 14.2: If M is complete, then M is non-extendible.

We give M a metric space structure by letting $d_M(p, q)$ be the infimum of lengths of all curves joining p and q. (This is the same as the original metric space structure.)

Theorem 14.3 (Hopf-Rinow Theorem)**:** thm:hopf-rinow Let M be a Riemannian manifold and let $p \in M$. Then the following are equivalent.

1. $\exp_p$ is defined on all T_pM.
2. The closed and bounded sets of M are compact.
3. M is complete as a metric space.
4. M is geodesically complete.

5. There exists a sequence of compact subsets $K_n \subseteq M$, $K_n \subset K_{n+1}$ such that if $q_n \notin K_n$ then $d(p, q_n) \to \infty$.

Any of these statements implies

6. For any $q \in M$ there exists a geodesic γ joining p and q with $\ell(\gamma) = d(p, q)$.

Lecture 15
Tue. 10/30/12

Last time we discussed the Hopf-Rinow Theorem, which says that different notions of completeness are all equivalent. Today we'll talk about the Hadamard Theorem.

§1 Hadamard Theorem

We'd like to see what the exponential map does to length. Recall that this question was related to the curvature of the space (see Section 11.1).

First we make an observation. Suppose (M, g) is a Riemannian manifold and J is a Jacobi field along a geodesic γ. Then the Jacobi equation

$$J'' + R(\gamma', J)\gamma' = 0$$

holds. Consider $f = \langle J, J \rangle$. We have

$$\begin{aligned} f &= \langle J, J \rangle \\ f' &= 2 \langle J', J \rangle \\ f'' &= 2 \langle J'', J \rangle + 2 \langle J', J' \rangle . \end{aligned}$$

Substituting in the Jacobi equation, we get

$$f'' = -2 \langle R(\gamma', J)\gamma', J \rangle + 2 \langle J', J' \rangle$$

Assume that $J \perp \gamma'$ (the component in the tangential component is always linear anyway), $J(0) = 0$, and the geodesic has unit speed: $|\gamma'| = 1$. Then

$$eq : 965 - 15 - 1 f'' = -2K(\gamma', J)|J|^2 + 2|J'|^2. \tag{30}$$

Consider the exponential map $\exp_p$ at some point v. Consider the parameterized surface $F(s, t) = sv(t)$ on T_pM where v is a curve with $v(0) = v$. Now the derivative in the t-direction forms a Jacobi field:

$$J(s_0) = \left.\frac{\partial}{\partial t}\right|_{t=0} \left(\exp_p F(s_0, t)\right) .$$

Suppose $v(0) = v_0$ and $v'(0) = w$. Then the above becomes

$$eq:965-15-2 J(s_0) = \frac{\partial}{\partial t}\bigg|_{t=0} \left(\exp_p F(s_0,t)\right) = d\exp_p\left(s_0 \frac{dv}{dt}\right) = d\exp_p(s_0 w). \tag{31}$$

We've reduced the problem of finding $|d\exp_p(w)|$ to finding the length of the vectors in the Jacobi field. Since the Jacobi field is related to curvature, this will tell us how much the exponential map distorts depending on the curvature. Letting $h = |J|$, $f = |J|^2$, we have

$$h(0) = 0$$
$$h'(0) = \frac{f'}{2\sqrt{f}} = \frac{2\langle J', J\rangle(0)}{2|J|(0)}.$$

(Implicitly, we mean $h'(0) = \lim_{v\to 0} \frac{2\langle J', J\rangle(v)}{2|J|(v)}$.) From (30), if $K \le 0$ we get

$$f'' \ge 2|J'|^2.$$

Lemma 15.1: If M is a Riemannian manifold with curvature $k \le 0$, for $w \in T_v(T_pM)$, we have

$$|d\exp_p(w)| \ge |w|.$$

This means the geodesics are expanding.

We write the proof for 2 dimensions. The proof in general is similar.

Proof. Fixing t in $F(s,t) = sv(t)$, we have

$$\frac{\partial}{\partial t} F = J_{\mathbb{R}^n}.$$

We have $J_{\mathbb{R}^n}(0) = 0$.Assume w is orthogonal. Define J as in (31) with $v(0) = v$, $v'(0) = w$. Because w is orthogonal, the Jacobi field is orthogonal.

In 2 dimensions, we can write $J = jE$. We already know that $J' = j'E$ and $J'' = j''E = -kjE$. If $j(0) = 0$ and $j'(0) = \frac{|w|}{|v|}$, $J'' \ge 0$ and j' is growing. This implies that $j' \ge \frac{|w|}{|v|}$.

Thus j is growing at least linearly, and we have

$$|d\exp_p(w)| = j(|v|) \ge \frac{|w|}{|v|}|v| = |w|.$$

This was for w orthogonal. In the radial direction the exponential map preserves the norm. Thus we get that $\exp_p$ is locally expanding. □

In the general case, consider $h = |J| = \sqrt{f}$ and get a differential inequality.

If (M,g) is a complete manifold, i.e. $\exp_p : T_pM \to M$ is defined on all of T_pM, then Hadamard's Theorem says the following.

Theorem 15.2 (Hadamard)**:** thm:hadamard If (M,g) is complete and $K \le 0$, then $\exp_p : T_pM \to M$ is a covering map.

Corollary 15.3: cor:hadamard M is complete and simply connected with $K \leq 0$, then M is diffeomorphic to $\mathbb{R}^n$ and $\exp_p : T_pM \to M$ is a diffeomorphism.

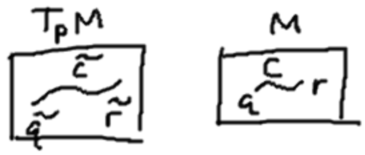
In fact, $\exp_p : T_pM$ is distance non-decreasing.

Proof of Corollary 15.3. Given Hadamard's Theorem, if M is simply connected, the covering space must be the same as the space itself, so $T_pM \to M$ is a diffeomorphism.

Let $\tilde{q}, \tilde{r} \in T_pM$, and $q, r \in M$ be their images in M. We show that

$$d_M(q,r) \geq |\tilde{q} - \tilde{r}|.$$

Let c be a curve from q to r. We can pull it back by the exponential map to get a curve $\tilde{c}$ from $\tilde{q}$ to $\tilde{r}$.



Suppose c is defined on $[a,b]$. We have

$$|\tilde{q} - \tilde{r}| \leq \text{length}(\tilde{c}) = \int_a^b |\tilde{c}'|\, ds \leq \int_a^b |c'|\, ds.$$

Taking the infimum over all c, we get the desired inequality. □

We will be pretty informal in the following. For details see [3, p. 149–151].

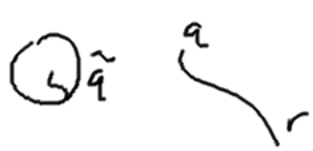
Proof of Theorem 15.2. We show that $\exp_p$ is a covering map. One way to show a map is a covering map is to show that it has the path-lifting property. The fact that $|d\exp_p(w)| \geq w$ gives that the derivative at any point in the tangent space is 1-to-1, so $\exp_p$ locally a diffeomorphism.

$\exp_p$ is onto by the Hopf-Rinow Theorem: In a complete manifold any pair of points can be joined by a geodesic, i.e. any other point is in the image of exponential map at p.

Given $q, r \in M$, we can find a neighborhood around $\tilde{q}$ such that a curve in that neighborhood is at mapped to a little piece of the curve c in M starting at q. Using $|d\exp_p(w)| \geq |w|$ we get that the length of the little curve at $\tilde{q}$ is less than or equal to the length of the curve at q:

$$\int_a^b |\tilde{c}'| \leq \int_a^b |c'|$$

Now go to the endpoint of the little curve and continue the process.



Completeness of M implies that the exponential map is onto. We'veused $|d\exp_p(w)| \geq |w|$ weakly to say it's a locally a diffeomorphism. We use it strongly to say it doesn't wander off to infinity: The lifted curve lies inside something compact by the inequality, so we can continue all the way to the end. (This part of the argument goes through if we just assume $|d\exp_p(w)| \geq c|w|$.)

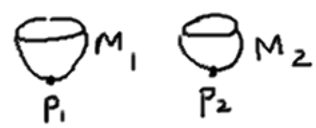
The only choice involved is the preimage $\widetilde{q}$ of q; then the curve is uniquely given.

The ODE for the geodesic extending c has a solution for all $\mathbb{R}$ if it doesn't go off to infinity. The inequality ensures that the lifted curve doesn't wander off to infinity. This shows the path-lifting property; hence $\exp_p$ is a covering map. □

§2 Constant curvature

Now we'll talk a little bit about constant curvature.

Let M_1^n, M_2^n be complete, simply connected manifolds with the same constant curvature. We'll prove next time that there is an isometry from M_1 to M_2. (Recall that an isometry is a metric-preserving isomorphism.)



We construct the isometry $I : M_1 \to M_2$ as follows. Take $p_1 \in M_1$ and $p_2 \in M_2$. In a neigborhood of p_1, we have a map

$$\exp_{p_1}^{-1} : M_1 \to T_{p_1}M_1.$$

We let $i : T_{p_1}M \to T_{p_2}M$ be any isometry taking 0 to 0. Now define

$$I = \exp_{p_2} \circ i \circ \exp_{p_1}^{-1}.$$

Lecture 16
Tue. 11/6/12

Colding is in Sweden today, so Bill Minicozzi is lecturing.

Yesterday when I was at the airport, I picked up a copy of Boston magazine. Harvard is now the second best university in Cambridge. In case you're wondering, MIT is the best.

This week we're going to do two things. We will

1. introduce hyperbolic space and
2. understand the classification of spaces of constant curvature.

There are many different notions of curvature: we could talk about sectional curvature, Ricci curvature, and scalar curvature (and others). Constant scalar curvature is the loosest condition, and constant sectional curvature is the strictest condition.

- Scalar curvature: A theorem of Rick-Shane says that given any closed manifold (compact without boundary), by just making conformal changes, we can make it a manifold of constant scalar curvature. In dimensions other than in dimension 2, we can get negative constant scalar curvature.
- Ricci curvature: A manifold with constant Ricci curvature is called an Einstein manifold. The Ricci cuvature is constant if the Ricci curvature is a constant multiple of the metric; this gives the **Einstein equation**. Constant Ricci curvature says quite a bit; in low dimensions it says a lot.
- Sectional curvature: Manifolds with constant sectional curvature are called **space forms**. We'll discuss these today.

The sectional curvature can be negative, 0, or positive. By scaling we can assume $\kappa = -1, 0, 1$. (For instance, if the curvature is 10, we can scale by $\frac{1}{\sqrt{10}}$ to make the curvature 1. A sphere of radius 1 has curvature 1; if we make sphere larger, then the curvature decreases.) In these 3 cases we have the following spaces.

1. $\kappa = -1$: Hyperbolic space H^n.
2. $\kappa = 0$: Euclidean space $\mathbb{R}^n$.
3. $\kappa = -1$: Sphere $\mathbb{S}^n$.

Hyperbolic space is the new manifold, which we haven't talked about. On Thursday we'll show that every complete simply connected manifold of constant curvature must be one of these: $H^n, \mathbb{R}^n$, or $\mathbb{S}^n$.

Of course, the manifold doesn't haven't to be simply connected. We could quotient out $H^n, \mathbb{R}^n$, or $\mathbb{S}^n$ by the action of a group of isometries. Then we get a manifold locally isometric to one of these spaces, but not globally diffeomorphic.

For instance, if we quotient the plane $\mathbb{R}^2$ by a translation we get a cylinder. If we quotient again by another translation, we get a torus, also of 0 sectional curvature. We can quotient different spaces by different lattices, and the classification of spaces becomes a question about group theory. Quotienting by hyperbolic groups is much more complicated. For the sphere, we have to quotient by a finite group because there is no fixed point free infinite group action. The resulting space has finite fundamental group. The study of quotient spaces here becomes the study of subgroups of the orthogonal group.

That's the background. Let's do math now.

§1 Conformal metrics

Definition 16.1: Let (M, g) be a Riemannian manifold. We say that a (M, gf) is a **conformal change of metric** if f is a constant depending only on the point, $f = f(p)$.

We say that f and gf are conformally related.

Think of g as a symmetric 2-tensor at each point. A conformal change allows us to multiply g by a constant at each point. This is one way of changing the metric.

Suppose we want to change the metric from g to $\overline{g}$. We need to make sure that $\overline{g}$ is still positive definite. One way is to add another positive definite form to g. We see that there are lots of ways to change the metric.

The space of $n \times n$ matrices has dimension n^2. The space of symmetric $n \times n$ matrices still has dimension on the order of n^2. A conformal change of metric gives us a comparatively small allowable space of changes, just a 1-dimensional space at each point. We wouldn't expect it to generate too many different metrics.

Suppose we fix a metric and look at all metrics conformal to it. In dimension 2, this one metric generates everything; every metric is conformal to every other metric locally.

Definition 16.2: df:conf-flat A metric is **conformally flat** if $g_{ij} = F^{-2}\delta_{ij}$ for some F.

All constant metrics are conformally flat. This is not an accident. Not every metric is conformally flat. In high dimensions, we can build a tensor built out of the curvature, called the Weyl tensor. It measures the obstruction to being locally conformally flat. This isn't quite true in dimension 3, though; we have to use the Bott tensor. If the tensor vanishes then the manifold is locally conformally flat.

Note that different authors may let the constant in Definition 16.2 depend differently F. I use F^{-2} to match Do Carmo's notation. I would prefer not to use F^{-2}. It is more natural to use e^F; this is also automatically positive. Set $f = \ln F$; this will come up in our calculations.

Our first task is to compute Cristoffel symbols and curvature tensors for a conformal change of metric.

1.1 Compute Γ_{ij}^k's

Recall the formula

$$eq:787-16-1\Gamma_{ij}^k = \frac{1}{2}\sum_m \left(g_{jm,i} + g_{mi,j} - g_{ij,m}\right) g^{mk} \tag{32}$$

where $g_{jm,i}$ denotes $\frac{\partial g_{jm}}{\partial x_i}$ and g^{mk} denotes the (m, k) entry in the inverse matrix of the (g_{ij}).

When $g_{ij} = F^{-2}\delta_{ij}$, we compute

$$g_{ij,k} = -2F^{-3}\frac{\partial F}{\partial x_k}\delta_{ij} = -2\left(F^{-1}\frac{\partial F}{\partial x_k}\right)F^{-2}\delta_{ij} = -2f_k g_{ij}$$

where $f_k = \frac{\partial \ln F}{\partial x_k} = F^{-1}\frac{\partial F}{\partial x_k}$.

We plug in this formula everywhere in (32):

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}\sum_m \left(g_{jm,i} + g_{mi,j} - g_{ij,m}\right) g^{mk} \\ &= -\sum_m \left(f_i g_{jm} + f_j g_{mi} - f_m g_{ij}\right) g^{mk} \\ &= -f_i\delta_{jk} - f_j\delta_{ik} + f_k\delta_{ij} \qquad\qquad \sum_m g_{jm}g^{mk} = \delta_{jk}.\end{aligned}$$

(A matrix times its inverse equals the identity.) Let's check that the last equality is above, on the third terms: $\sum_m f_m g_{ij} g^{mk} = f_k\delta_{ij}$. We have $g^{mk} = F^2\delta_{mk}$ so

$$\begin{aligned}\sum_m f_m g_{ij} g^{mk} &= \sum_m f_m(F^{-2}\delta_{ij})(F^2\delta_{mk}) \\ &= f_k\delta_{ij},\end{aligned}$$

as expected. We consider several cases.

- i, j, k distinct: $\Gamma_{ij}^k = 0$. We only have to worry about if exactly 2 are the same, or all 3 are the same.

For what I write next there is no summation convention. (Usually if same index appears twice, we sum over over it. Here we're going to write formulas containing repeated indices over it, but I don't want to sum over it.)

- $i = j = k$: $\Gamma_{ii}^i = -f_i$.
- $i = j \neq k$: $\Gamma_{ii}^k = 0 - 0 + f_k = f_k$.
- $i = k \neq j$: $\Gamma_{ij}^i = -f_j$.
- By symmetry, $\Gamma_{ji}^i = \Gamma_{ij}^i = -f_j$.

It's good to go through all the calculations once. This not something you do again as practicing geometer. Once you've seen how the calculations go, you never need to compute the Γ_{ij}^k again. When I have to make a conformal change of coordinates to get a metric with certain quantities, I go to my trusty *Lectures in Differential Geometry* and thumb through it until I find the formula. But you have to do this once for yourself before you're allowed to look it up.

Note that a conformal change in metric preserves orthogonality. It stretches the lengths of all vectors in a tangent space by same amount. A conformal change preserves angles, and just stretches distances. The most obvious conformal change is to dilate the whole manifold by a constant. Any conformal map is like this at a point.

A map is conformal if its effect on the metric (i.e., the pullback) is a conformal change. The map stretches lengths and preserve angles.

1.2 Curvature tensor

Our next task is to use the Christoffel symbols to compute the curvature tensor. Then we will use the curvature tensor to get the sectional curvature.

Again, we don't use summation notation in what follows. We have

$$\begin{aligned} R_{ijij} &= \sum_\ell R^\ell_{iji} g_{\ell j} \\ &= R^j_{iji} g_{jj} && g_{\ell j} = 0 \text{ for } \ell \neq j \\ &= F^{-2} R^j_{iji} \\ &= F^{-2}\left[\left(\sum_\ell \Gamma^\ell_{ii}\Gamma^j_{j\ell} - \Gamma^\ell_{ji}\Gamma^j_{i\ell}\right) + \partial_j \Gamma^j_{ii} - \partial_i \Gamma^j_{ji}\right] \end{aligned}$$

We don't care to compute $R_{ijk\ell}$ in general; we just compute R_{ijij} so we can get the sectional curvature.

We also need derivatives of the Γ's, so let's record what they are. We have

$$\begin{aligned} \partial_j \Gamma^j_{ii} &= f_{jj} := \frac{\partial^2 f}{\partial {x_j}^2} \\ \partial_i \Gamma^j_{ji} &= -f_{ii}. \end{aligned}$$

We have

$$\partial_j \Gamma^j_{ii} - \partial_i \Gamma^j_{ji} = F^{-2}(f_{jj} + f_{ii}).$$

We now split the sum into 3 cases. We only need to consider $i \neq j$, because it is 0 otherwise. We obtain

$$R_{ijij} = F^{-2}(f_{jj} + f_{ii}) + \underbrace{F^{-2}(\Gamma^i_{ii}\Gamma^j_{ji} - \Gamma^i_{ji}\Gamma^j_{ii})}_{\ell=i} + \underbrace{F^{-2}(\Gamma^j_{ii}\Gamma^j_{jj} - (\Gamma^j_{ji})^2)}_{\ell=j} + \underbrace{F^{-2}\sum_{\ell\neq i,j}(\Gamma^\ell_{ii}\Gamma^j_{j\ell} - \Gamma^\ell_{ji}\Gamma^j_{i\ell})}_{\ell\neq i,j}.$$

$$\begin{aligned} &= F^{-2}\left[(f_{ii} + f_{jj}) + [\cancel{f_i}^2 - \cancel{(-f_j)(f_j)}] + [\cancel{f_j(-f_j)} - \cancel{f_i}^2] + \sum_{\ell\neq i,j}[f_\ell(-f_\ell) - 0]\right] \\ &= F^{-1}\left[(f_{ii} + f_{jj}) - \sum_{\ell\neq i,j}(f_\ell)^2\right]. \end{aligned}$$

The good news is that this agrees with what's in my notes! We now have

$$eq: 787-16-2 \kappa_{ij} = \frac{R_{ijij}}{\det(g_{ij})} = \frac{R_{ijij}}{g_{ii}g_{jj}} = \frac{F^{-2}\left(f_{ii} + f_{jj} - \sum_{\ell\neq i,j} f_\ell^2\right)}{F^{-4}} = F^2\left(f_{ii} + f_{jj} - \sum_{\ell\neq i,j} f_\ell^2\right). \quad (33)$$

This formula is valid for any conformal metric. We've now computed the sectional curvature for any conformal metric in terms of the original metric.

The highest-order term $f_{ii} + f_{jj}$ looks like a Laplacian. The lower-order term looks like the gradient squared of the log of the function.

Example 16.3: Consider the case of $\mathbb{R}^2$. At a point there is only 1 possible sectional curvature. There are no $\ell \neq i, j$ terms. The curvature is the Gauss curvature, $F^2(f_{ii} + f_{jj})$, which is really a Laplacian, i.e., the trace of the hessian, the sum of the second derivatives "down the diagonal."

Let's now specialize to hyperbolic space.

§2 Hyperbolic metric

There are two conformal models of hyperbolic space: upper half space and the unit disc. The easiest to compute for us is the upper half space: F is a function of x_n. In the unit disc model we would have to use polar coordinates.

Define upper half-space by

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_n > 0\}$$

and give it the hyperbolic metric given by

$$g_{ij} = \frac{\delta_{ij}}{x_n^2} = F^{-2}\delta_{ij} \quad \text{where} \quad F = x_n,\ f = \ln x_n.$$

We will check that this metric is complete. To do this, we take a straight line going out. the length is constant multiple $\frac{1}{x_n}$ of the Euclidean length. A positive number times an infinite length is infinite.

What if we reach $x_n = 0$ and the length of curve is finite? If so, then the space would not be complete. If we take a line straight down, then is complete: to find the length we compute the integral of $\frac{1}{x_n}$. The length is $-\ln x_n \to \infty$ as $x_n \to 0$. However, this is just one path down. Need to check that every path down takes infinitely long. The same check works for paths that go up infinitely.

We make these computations rigorous, but first we check that H^n does indeed have constant curvature.

2.1 H^n has constant curvature

We calculate the sectional curvature of H^n using (33). For $i \neq n$ we have $f_i = 0$. We have $f_n = \frac{1}{x_n}$. For $i \neq n$ we have $f_{ii} = 0$. We have $f_{nn} = -\frac{1}{x_n^2}$. Consider several cases.

- $i, j \neq n$: $\kappa_{ij} = x_n^2\left[0 + 0 - \left(\frac{1}{x_n}\right)^2\right] = -1$. This is a very happy result because we introduced H^n as having curvature -1.
- $j = n$: $\kappa_{in} = x_n^2\left[0 - \frac{1}{x_n^2} - 0\right] = -1$.
- $i = n$: $\kappa_{nj} = x_n^2\left[-\frac{1}{x_n^2} + 0 - 0\right] = -1$.

All possible planes have sectional curvature -1. In the case of the hyperbolic plane H^2, we don't even have to worry about the κ_{ij} case, and this is quicker to see.

Next we check the completeness of hyperbolic space.

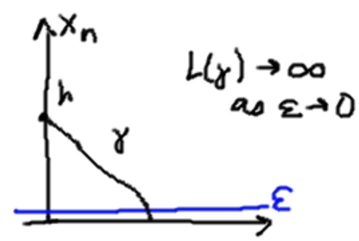
2.2 Hyperbolic space is complete

Translation perpendicular to x_n leaves lengths unchanged; this is essentially just a choice of coordinates. Thus, we only need to show that any curve from a point on the x_n-axis to infinity has infinite length.

Consider the region bounded vertically by $\varepsilon < x_n < \frac{1}{\varepsilon}$ and horizontally by $\sqrt{x_1^2 + \cdots + x_{n-1}^2} \leq \left(\frac{1}{\varepsilon}\right)^2$. (Think of this as a "soup can;" it is round at the edges. In two dimensions it is just a rectangle.) Any divergent curve must hit one of these boundaries.

1. First suppose the curve hits one of the sides. We look how long any curve to the side is. The Euclidean length is at least the distance from the axis to the side which is at least $\frac{1}{\varepsilon^2}$. But the metric is always at least ε times the Euclidean one, so the hyperbolic length is at least $\varepsilon\left(\frac{1}{\varepsilon^2}\right) = \frac{1}{\varepsilon}$:
$$L = \int \sqrt{g(\gamma', \gamma')} = \int F^{-1}|\gamma'|_{\text{Euclidean length}} \geq \varepsilon\left(\frac{1}{\varepsilon^2}\right) = \frac{1}{\varepsilon}.$$
The length is at least $\frac{1}{\varepsilon}$. If $\varepsilon \to 0$ this goes to infinity so we're good.

2. We just have to worry about the curve hitting the top or bottom of the soup can. The two cases are basically symmetric; I'll do one and you'll see the other. Suppose a curve goes south.



Write
$$\gamma(t) = (x_1(t), \ldots, x_n(t)).$$
We have
$$\begin{aligned} L &= \int \frac{1}{x_n(t)} \sqrt{(x_1')^2 + \cdots + (x_n')^2}\, dt \\ &\geq \int \frac{\sqrt{(x_n')^2}}{x_n}\, dt = \int \frac{|x_n'|}{x_n}\, dt \\ &\geq |\text{change in } \ln(x_n)| \to \infty \text{ as } \varepsilon \to 0. \end{aligned}$$
Note that "wiggling" back and forth horizontally only make length bigger, and we only have to show curves go straight up and down have infinite length. We just use the fundamental theorem of calculus! (Note we have inequality because the curve might go up and down; i.e., x_n may not be monotonic.)

This completes the proof.

§3 Hyperbolic geodesics and isometries

We'll come back to isometries. Let's find the geodesics in hyperbolic space.

First, vertical lines give geodesics. The geodesic equation is

$$x_k'' + \sum_{i,j} \Gamma_{ij}^k x_i' x_j' = 0.$$

First consider the special case where $x_1, \ldots, x_{n-1}$ are constant. Letting $x_n = h(t)$, the equation becomes

$$0 = h'' + \Gamma_{nn}^n (h')^2 = h'' + (h')^2 \left(-\frac{1}{h}\right).$$

(note in the hyperbolic case $\Gamma_{nn}^n = -\partial_n \ln x_n = -\frac{1}{x_n}$) or

$$h''h - (h')^2 = 0.$$

One obvious solution is $h(t) = e^t$. Note $h(t) = ae^t$ also works for any $a > 0$, and $h(t) = e^{-t}$ also works. This suggests that the general solution is hyperbolic functions. Let's stop there. I'll see you Thursday.

Lecture 17
Thu. 11/8/12

Today we'll finish talking about space forms. We'll show that the 3 spaces of constant curvature we already know are the only simply connected spaces of constant curvature. Here is the main theorem.

Theorem 17.1: thm:space-forms Let M^n be a complete simply connected manifold of constant curvature κ.

1. If $\kappa = -1$, $M \cong H^n$ (M is isometric to hyperbolic space).
2. If $\kappa = 0$, $M \cong \mathbb{R}^n$.
3. If $\kappa = -1$, $M \cong S^n$.

As we've said, we can take care of different κ by scaling.

If M is not simply connected, then the universal cover is simply connected, and is one of H^n, $\mathbb{R}^n$, and S^n. Let's consider some examples.

1. $\kappa = 1$: The cylinder $S^1 \times \mathbb{R}^{n-1}$. This is not $\mathbb{R}^n$. However, if you take the universal cover, i.e., unroll the cylinder, you do get $\mathbb{R}^n$.

2. $\kappa = -1$: We can have hyperbolic surfaces of genus $g > 1$.

   For a compact manifold (closed without boundary), there is just one number that matters topologically, and that is the genus. The sphere has genus 0, the torus has genus 1, and all compact complete manifolds of genus 2 with constant curvature are hyperbolic.

   There are tons of things with constant negative curvature.

3. $\kappa = 0$: $\mathbb{RP}^n$, or any quotient of S^n by a linear group action without fixed points. $\mathbb{RP}^n$ is the space of rays through origin (i.e., a pair of antipodal points of S^n). We identify east and west, etc. In the first week of class, we showed this is a manifold; it is S^n modulo the antipodal transformation. The group action is $\mathbb{Z}/2$ because if you flip twice, you get back where you started. Note $\mathbb{Z}/2$ is part of the orthogonal group.

   More generally, we can take any finite subset of the orthogonal group with no fixed points. The fundamental group is the group you quotiented by. This describes all of the manifolds with constant positive curvature. Milnor listed all subgroups of orthogonal group that can act, and put them in categories.

   As another example, quotienting by $\mathbb{Z}/p$ we get lens spaces. The fundamental group is $\mathbb{Z}/p$.

Note that simply connected implies oriented, but quotients may not be orientable. For instance $\mathbb{RP}^n$ in certain dimensions is unorientable, even though it has flat metric. The fundamental group is $\mathbb{Z}/2$ in those dimensions.

This is not the end of the story. Lots interesting things are still going on in 3-manifold theory. There is not too much going on in the study of flat or positively curved surfaces. Almost the whole field concerned with the study of hyperbolic manifolds. After geometrization, there are 8 possible manifold geometries. The hyperbolic is the most interesting. It is basically a group question (what groups can act on hyperbolic space?), and the groups can be extremely complicated.

§1 Curvature $\kappa = 0, -1$

We will prove Theorem 17.1 for $\kappa = 0, -1$ at same time. The case for $\kappa = 1$ is different. For $\kappa = 0, -1$, if the space is simply connected, then $\exp_p$ is a global diffeomorphism by Hadamard's Theorem, and we have a natural map to start working with. In the case $\kappa = -1$ we can also consider a map from the tangent space to the model space H^n. By taking the inverse of one and composing, we have a map from the manifold to H^n.

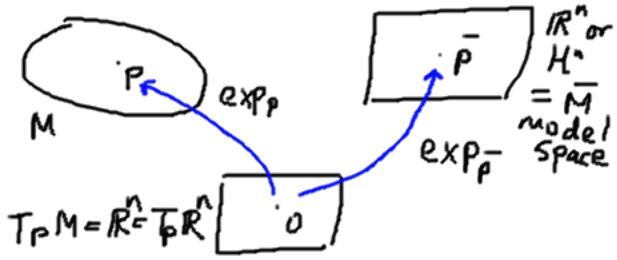
In the case of S^n, $\exp_p$ will not be globally defined.

To show the isometry, we actually construct a isometry. In the cases $\kappa = 0, -1$, nat cand right in front of us, and we just have to show it works. In the case $\kappa = 1$, we have to go through more work to find it.

We'll use a technical result and defer its proof.

We write the proof for $\mathbb{R}^n$; the same proof holds for H^n. Let $p \in M$ and $\overline{p} \in \mathbb{R}^n$ (or H^n). We identify $T_p\mathbb{R}^n = \mathbb{R}^n = T_pM$. We have the exponential maps are globally well-defined invertible diffeomorphisms by Hadamard's Theorem 15.2:

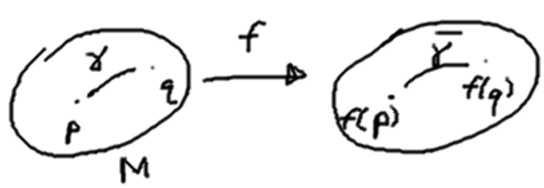
$$\exp_M : T_pM \to M, \qquad \exp_{\overline{M}} : T_p\mathbb{R}^n \to \mathbb{R}^n.$$



1. Set $f = \exp_M \circ \exp_{\overline{M}}^{-1} : \overline{M} \to M$. We have $f(\overline{p}) = p$. Both exponential maps (and their inverses) are diffeomorphisms, so f is a local diffeomorphism.

2. By a theorem of Cartan, f is a local isometry. We'll come back to proving this (it is somewhat of a pain).

   Theorem 17.2 (Cartan)**:** thm:cartan Suppose $f : M \to \overline{M}$ is a local diffeomorphism, $f(p) = \overline{p}$, and $df_p = \text{id}$ (i.e., we identify $T_pM = T_{\overline{p}}\overline{M}$. Let $\gamma : [0, \ell] \to M$ be a geodesic from p to q, let $\overline{\gamma} = f \circ \gamma$, let P_t be parallel transport from p along γ, and let $\overline{P}_t$ be the parallel transport from $\overline{p}$ to $\overline{\gamma}(t)$. Define the map $\phi_\ell : T_qM \to T_{f(q)}\overline{M}$ on tangent spaces by the following.
   $$\phi_\ell = \overline{P}_\ell \circ P_\ell^{-1}.$$
   If
   $$\langle R(x, y)u, v\rangle = \left\langle \overline{R}(\phi_\ell(x), \phi_\ell(y))\phi_\ell(u), \phi_\ell(v)\right\rangle$$
   for all x, y, u, v, and all q in a neighborhood of p, then f is a local isometry at p.



   (When we parallel transport back from q to p, the vector now automatically lives in $T_{\overline{p}}\overline{M}$ because df_p is the identity.)

   We can use Cartan's Theorem because the curvature of M and $\overline{M}$ are the same. Now ϕ_ℓ is an isometry because it is built out of isomotries: parallel transport is an isometry.

Thus the inner product between x, y is the same as between $\phi_t(x)$, $\phi_t(y)$. Thus Cartan's Theorem is satisfied, and f is a local isometry. Now we just need to show it's a global isometry.

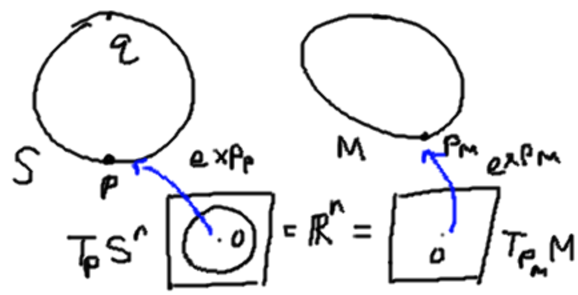
Note that from step 1, f is a local diffeomorphism follows from step 1.

3. By Lemma 3.3 in Chapter 7 (any local diffeomorphism from a complete Riemannian manifold with the property $|df_p(v)| \geq |v|$ is a covering map), f is a covering map. Now the fact that M is simply connected implies f is a global isometry.

The only thing we haven't proven is the theorem of Cartan. See Section 3.

§2 Curvature $\kappa = 1$

In the case $\kappa = 1$, the exponential map isn't a global diffeomorphism, we have to cook something up. Let $p \in S^n$ and let q be the antipodal point to p.



Identify $T_pS^n = \mathbb{R}^n = T_{p_M}M$. We have exponential maps

$$\exp_p : T_pS^n \dashrightarrow S^n \backslash \{q\}$$

where the map is only defined on a ball B_π of radius π. We have

$$\exp_M : T_{p_M}M \to M.$$

Set

$$f = \exp_M \circ \exp_p^{-1} : S^n \backslash q \to M.$$

As before, Cartan's theorem only says that f is a local isometry.

However, we don't get a map on all of S^n!

We can pick another pair of antipodal points $\overline{p}, \overline{q} \in S^n$ and define $\overline{f} : (S^n \backslash \overline{q}) \to M$.

Each map is defined on the sphere minus 2 antipodal points. We need to show that they agree on the intersection, the sphere minus 4 points.

Let's show that f and $\overline{f}$ agree where both are defined. Assuming this, we get a global map (from gluing f and $\overline{f}$) $f \vee \overline{f} : S^n \to M$ that is a local isometry on a compact manifold. Again by Lemma 3.3 in Chapter 7, it is a covering map. Since $f \vee \overline{f}$ is a covering map, it is simply connected, so it is a global isometry.

It remains to show that $f, \overline{f}$ agree on overlap.

Lemma 17.3 (Monodromy lemma)**:** Suppose that $f, \overline{f} : M \to N$ are local isometries, and M is connected. Suppose that

$$f(p) = \overline{f}(p)$$
$$df_p = d\overline{f}_p$$

for at least one point p. Then $f \equiv \overline{f}$.

In other words, if we have a point where f and $\overline{f}$ agree to first order, then they agree completely.

This is going to be one of those "open and closed" arguments: The set where $f, \overline{f}$ agree is open and closed, is nonempty, so whole space.

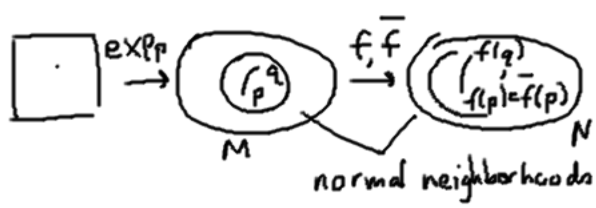
Note the lemma fails if we don't assume $df_p = d\overline{f}_p$. In $\mathbb{R}^n$, the identity map and rotation by 90° are local isometries fixing the origin that don't agree. However, 2 rotations that fix the origin and are the identity on the (tangent space of the) origin must agree.

Proof. Let

$$S := \left\{q : f(q) = \overline{f}(q) \text{ and } df(q) = d\overline{f}(q)\right\}.$$

First, S is closed because of continuity (f and df continuous). The intersection of 2 closed subsets is closed. Note $p \in S$ by assumption.

The tricky part is showing S is open. We'll do this by a picture. S is open since $f = \overline{f}$ in a neighborhood about p. Consider the exponential map $\exp_p$ around p.



Take normal neighborhoods of both p and $f(p)$. In each of these neighborhoods $\exp_p$ is a diffeomorphism. Geodesics are unique.

Take q in the neighborhood and let γ be the unique geodesic between p and q; it is minimizing. There is some minimizing geodesic $f(p)$ to $f(q)$.

1. f and $\overline{f}$ are local isometries, so they send geodesics to geodesics. They must send the geodesic γ to some geodesic in the image. (Warning: we don't know the endpoints are the same yet, because *a priori* maybe $f(q) \neq \overline{f}(q)$.)

2. But these geodesics satisfy the same initial conditions: Because $p \in S$, $df_p(\gamma'(0)) = d\overline{f}_p(\gamma'(0))$. Hence $f \circ \gamma$, $\overline{f} \circ \gamma$ are geodesics starting at the same point, with same derivative at 0. So they are the same.

Thus in this neighborhood f and $\overline{f}$ agree identically. Cetainly they agree to first order. This proves this monodromy lemma. □

We're done with the proof of the main theorem modulo the proof of Cartan's theorem.

§3 Cartan's theorem

sec:cartan We prove Cartan's Theorem 17.2.

Proof. To show f is a local isometry, we have to show that given any $v \in T_qM$, then

$$eq:787-17-3 |df_q(v)| = |v| \tag{34}$$

where the first is the norm on $T_{f(q)}\overline{M}$ and the RHS is the norm on T_qM. This says that the length after you apply df_q is the same as before. Because f preserves the length of all tangent vectors, it will be an isometry.

How do we compute $df_q(v)$?

Assume that $f = \exp_{\overline{p}} \circ \exp_p^{-1}$ as before. f is built out of 2 exponential maps. By the chain rule, the differential is the composition of 2 differentials of exponential maps (with one inverted). We have to figure the differential of an exponential map.

The differential of a exponential map is given by Jacobi fields. Why? A vector on a geodesic produces a 1-parameter family of geodesics, and the Jacobi field tells you how that 1-parameter family of geodesics changes. More precisely, Proposition 17.4 tells us how the Jacobi field relates to the differential of the exponential.

Given $v \in T_qM$, choose a Jacobi field J along γ so that

$$J(0) = 0 \qquad J(\ell) = v,$$

where ℓ is the length of γ. Given 2 tangent vectors at two ends of a geodesic in a normal neighborhood, there always exists a Jacobi field linking them. Choose an orthonormal frame $e_i(0)$ at T_pM and parallel transport it to get $e_i(t)$. We assume $e_n(t) = \gamma'$.

Now J will tell us the differential of the exponential map on γ. We'll get another $\overline{J}$ along $\overline{\gamma}$ that tell us what the Jacobi field is doing over there. We need to show $J = \overline{J}$, so that the differentials are the same.

Why sould these 2 Jacobi fields be the same? Because they satisfy the same initial conditions, and the same ODE. Why should they sat the same ODE? By hypothesis. The ODE for J has J'' and a curvature term. *The hypothesis tells us the curvatures are the same, so the Jacobi fields satisfy the same ODE.* By uniqueness of ODE's, the Jacobi fields are the same. This will tell us that the $d\exp$'s are the same, so if we compose the inverse of one with the other we get id, and (34) holds.

We formalize this argument.

Using the orthonormal frame, write

$$J(t) = \sum_{i=1}^n y_i(t) e_i(t).$$

The Jacobi equation tells us

$$y_j'' + \sum_{i=1}^n \langle R(e_n, e_i)e_n, e_j \rangle \, y_j = 0.$$

We get a 2nd order system of ODE's for y_j. Let $\phi_t(e_i(t)) = \overline{e}_i(t)$ along $\overline{\gamma}$; ϕ_t moves tangent vectors on γ to tangent vectors on $\overline{\gamma}$. Consider $\overline{J} := \phi_t(J)$. Note that ϕ_t takes γ' to $\overline{\gamma}'$. Both $\gamma, \overline{\gamma}$ are geodesics, and parallel transport preserves $\gamma', \overline{\gamma}'$.

We have

$$\overline{J} = \sum_{i=1}^{n} y_i(t)\overline{e}_i(t)$$

is also a Jacobi field. This is because it satisfies the same ODE (since the curvatures are the same by hypothesis), just with bars on top.

What else does this mean? We now relate Jacobi fields and $d\exp_p$. Corollary 2.5 in Chapter 5 says the following.

Proposition 17.4: pr:jacobi-dexp If $J(0) = 0$ then

$$J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0)).$$

If we differentiate a 1-parameter family of geodesics we get $J(t)$. How do we know which family we should use to get $d\exp_p v$? $d\exp_p v$ corresponds to how J changes at 0; it tells us how we're varying the family ("wedge") of geodesics.

The same proposition tells us

$$\overline{J}(t) = (d\exp_{\overline{p}})_{t\gamma'(0)}(t\overline{J}'(0)).$$

Now $df_q = (d\exp_{\overline{p}})_{\exp_p^{-1}(q)} \circ (d\exp_p)_q^{-1}$. Note that $\exp_p^{-1}(q) = \ell\gamma'(0)$. Hence the two equations for J and $\overline{J}$ imply

$$df_q(J(\ell)) = \overline{J}(\ell).$$

Now $J(\ell)$ and $\overline{J}(\ell)$ have the same norm. This is because we built $\overline{J}$ out of J by parallel transport (ϕ_t), and parallel transport preserves length. Now $J(\ell) = v$, so (34) holds.

This is what we wanted to show. □

Now that we've finished, let's think about why this work.

What really make tge theorem work is that we can compute the differential in terms of Jacobi fields. We needed the Jacobi fields to be the same. Why are they the same? Because the Jacobi equations are the same. The Jacobi equation is written using the curvature; if we know the curvature is the same, the Jacobi fields are the same.

Lecture 18
Tue. 11/13/12

Let (M, g) be a Riemannian manifold and c be a curve on M. If $F : [a, b] \times (-\varepsilon, \varepsilon) \to M$ is a parametrized surface with $c = F(\bullet, 0)$, then $\left.\frac{\partial F}{\partial t}\right|_{t=0} = V$ is a variational vector field along c corresponding to F.

Conversely every vector field along c is a variational vector field for some F: if V is a vector field along c, we can let

$$F(s,t) = \exp_{c(s)}(tV(s)) \tag{35}$$
$$F(s,0) = \exp_{c(s)}(0) = c(s).$$

For some ε, this is defined for all $t \in (-\varepsilon, \varepsilon)$ and all $s \in [a,b]$, and we have that $V = \frac{\partial F}{\partial t}\Big|_{t=0}$, as needed. (The proof is straightforward; see do Carmo [3, Prop. 9.2.2].)

§1 Energy

As before, let $c : [a,b] \to M$ be a curve.

Definition 18.1: The **length** of c is

$$L(c) = \int_a^b |c'|\, ds$$

and the **energy** of c is

$$E(c) = \int_a^b |c'|^2\, ds.$$

Proposition 18.2: We have

$$L(c)^2 \le (b-a)E(c)$$

with equality iff $|c'|$ is constant ("c' has constant speed").

Proof. By the Cauchy-Schwarz inequality,

$$L(c) = \int_a^b |c'|\, ds \le \left(\int_a^b |c'|^2\right)^{\frac{1}{2}} \left(\int_a^b 1^2\, ds\right)^{\frac{1}{2}} = \sqrt{E(c)}\sqrt{b-a}$$

with equality iff $|c'|$ is proportional to 1 everywhere, i.e. $|c'|$ is constant. □

In particular, a geodesic γ has constant speed so

$$L(\gamma)^2 = (b-a)E(\gamma).$$

If c is any curve and $\gamma : [a,b] \to M$ is a minimizing geodesic between $p := c(a)$ and $q := c(b)$, then

$$E(c) \ge (b-a)L(c)^2 \ge (b-a)d^2(p,q) = E(\gamma).$$

Thus we see that the minimizing geodesic has the minimal energy of all curves from p to q. Furthermore, if c has the minimal energy among all curves between a and b, then equality holds everywhere above and c must be a minimizing geodesic.

The length seems like a perfectly good quantity. What's the advantage of looking at the energy? We want a quantity that is minimized exactly when the curve is a geodesic, so we can apply calculus of variations to study geodesics.

We want to minimize the length. A (minimizing) geodesic has minimal length. However, if we reparametrize the geodesic, it still has minimal length, but it is no longer a geodesic if the speed is not constant.

The advantage of energy over length is that if we minimize the *energy*, we not only fix the length of the curve, we also fix the speed through the curve. If the curve speed up or slows down, then it would have greater energy.

> The energy of a curve $c : [a,b] \to M$ from p to q is minimized exactly when c is a minimizing geodesic from p to q.

Now we compute the first and second variations of energy.

§2 Variations of energy

Let (M, g) be any manifold with nondegenerate symmetric bilinear form. Let $F : [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a parametrized surface that is a variation of the central curve $F(s,0) = c(s)$. We can think of the energy as a function of s:

$$E(F(\bullet, t)) = \int_a^b \left| \frac{\partial^2 F}{\partial s^2} \right| ds.$$

Taking the derivative gives

$$\frac{d}{dt} E(F(\bullet, t)) = 2 \int_a^b \left\langle \frac{D}{\partial t} \frac{\partial F}{\partial s}, \frac{\partial F}{\partial s} \right\rangle ds.$$

We rewrite this purely in terms of the central curve c and the variational vector field corresponding to F along c. To do this, we need to change $\frac{D}{\partial t}$ to $\frac{D}{\partial s}$:

$$\begin{aligned} 2 \int_a^b \left\langle \frac{D}{\partial t} \frac{\partial F}{\partial s}, \frac{\partial F}{\partial s} \right\rangle ds &= 2 \int_a^b \left\langle \frac{D}{\partial s} \frac{\partial F}{\partial t}, \frac{\partial F}{\partial s} \right\rangle ds && \text{Prop. 6.7} \\ &= 2 \int_a^b \frac{d}{ds} \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial s} \right\rangle ds - 2 \int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{D}{\partial s} \frac{\partial F}{\partial s} \right\rangle ds \\ &= 2 \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial s} \right\rangle \Bigg|_{s=a}^{s=b} - 2 \int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{D}{\partial s} \frac{\partial F}{\partial s} \right\rangle ds. \end{aligned}$$

How does the energy change for curves near c; i.e. what is the derivative of energy? We have shown the following.

Proposition 18.3 (First variational formula)**:** pr:1st-var-E Let (M, g) be any manifold with nondegenerate symmetric bilinear form. Let $F : [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a parametrized surface that is a variation of the central curve $F(s,0) = c(s)$. Then

$$\frac{d}{dt} E(F(\bullet, t)) = 2 \left\langle V, c' \right\rangle \big|_{s=a}^{s=b} - 2 \int_a^b \left\langle V, c'' \right\rangle ds.$$

Consider the following two statements.

1. The variational vector field V satisfies $V(a) = V(b) = 0$.
2. On the parametrized surface, all of the curves start at the same point and end at the same point: $F(a,t) = c(a)$ and $F(b,t) = c(b)$.

If statement 2 holds, then the t derivative at a and b are 0, so statement 1 holds. Conversely, if statement 1 holds, then there exists a parametrized surface with variational vector field V satisfying statement 2: exponentiating as in (35) gives us curves that begin and end at the same point.

We're interested in comparing the energy of a curve and of "competing" curves. If the competing curves don't start and end at the same point, then they're not good competitors. Just by move moving the starting point in, we can trivially decrease the energy. Thus we restrict to competitors with the same starting point and endpoint. Then the first term in Proposition 18.3 is 0:

$$eq:965-18.2\frac{d}{dt}E(F(\bullet,t)) = -2\int_a^b \langle V, c''\rangle \; ds \tag{36}$$

We have the following.

Proposition 18.4: c is a geodesic iff, for any proper variation F of c (i.e., variation fixing the start and endpoints), $\frac{d}{dt}E(F(\bullet,t)) = 0$.

Proof. If c is geodesic then by (36) the variation is 0 because $c'' = 0$.

Conversely, suppose that we know that for proper variations F, $\frac{d}{dt}E(F(\bullet,t)) = 0$. We show c is a geodesic. Consider a cutoff function ϕ with support in (a,b) that is 1 on $(a+\delta, b-\delta)$ and 0 at a and b. Let $V = \phi c''$, and F be the associated proper variation. Then

$$\left.\frac{\partial}{\partial t}\right|_{t=0} F = \phi c'' \implies -2\int_a^b \phi |c''|^2 \, ds = 0.$$

Since $\phi \ge 0$ on $[a+\delta, b-\delta]$, we get $|c''| = 0$ on $[a+\delta, b-\delta]$; this works for all $\delta > 0$ so $c'' = 0$. $\square$

To summarize, we look at a variation fixing the endpoints. If the central curve is a geodesic, then is derivative of energy at the central curve is always 0. We also have the converse: If we have a curve so that the derivative of energy is 0 for all variations fixing the endpoints then that curve must be a geodesic.

Another way of saying this is the following.

> The geodesics are exactly the critical points of the energy function.

We see that energy is much better to work with than length. The parametrization of a curve doesn't matter for length, but it does matter for energy.

Another perspective is the following. For each curve $c : [a, b] \to M$ in the manifold we assign an energy. Consider a new space made of curves. This is an extremely large space; it is an infinite-dimensional manifold We have a function on this space, the energy of the curve. What are the critical points of this function, if we just look at curves with the same starting and endpoint? The critical points in this infinite-dimensional space are exactly the geodesics. A curve of curves is exactly a parameterized surface. Saying that c is a critical point is saying that if we take a curve (parameterized surface) containing c, then the derivative has to be 0.

§3 Second variation of energy

We are interested in computing the second derivative of a function when its first derivative is 0.

We are hence interested in computing the second derivative of the energy at geodesics. Given a manifold and a geodesic $c : [a, b] \to M$, we look at a 1-parameter family of curves where this is the central curve and the other curves start and end at the same point as c.

Thus we have a parametrized surface with central curve

$$F(\bullet, 0) = c.$$

First suppose that for each t, $\frac{\partial F}{\partial t} = 0$ at $s = a, b$. We compute $\frac{d}{d}t\Big|_{t=0} E$ by differentiating (36). Again we want to rewrite the expression using things that are defined on c (without anything in the t-direction)

$$\begin{aligned}
\frac{d^2}{dt^2}\bigg|_{t=0} E(F(\bullet, t)) &= \frac{d}{dt}\bigg|_{t=0}\left(\frac{d}{dt}E\right) \\
&= \frac{d}{dt}\bigg|_{t=0}\left(-2\int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{D}{\partial s}\frac{\partial F}{\partial s}\right\rangle ds\right) \\
&= -2\int_a^b \left\langle \frac{D}{\partial t}\frac{\partial F}{\partial t}, \cancel{\frac{D}{\partial s}\frac{\partial F}{\partial s}}\right\rangle ds - 2\int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial F}{\partial s}\right\rangle ds \quad \text{initial curve geodesic} \\
&= -2\int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial s}F\right\rangle ds
\end{aligned}$$

Recall that if we had a parametrized surface W, we can change the order of differentiation if we bring in the curvature (Proposition 8.5),

$$eq : 965 - 18.3 \frac{D}{\partial t}\frac{D}{\partial s}W = \frac{D}{\partial s}\frac{D}{\partial t}W + R\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right) w. \tag{37}$$

(We want to rewrite the expression with quantities defined at just this curve c, namely c, c' and V. It's find to have derivatives of V in the s-direction along the curve c. We don't want derivatives t-direction in our final expression because they have nothing to do with the central curve. The curvature is fine as long as curvature along c. A double derivative in t direction is troublesome, so it's good that the first integral vanished. We want to switch $\frac{D}{\partial s}$ and $\frac{D}{\partial t}$ so we can turn $\frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial F}{\partial s}$ into 2 covariant derivatives along c, as below.) Putting in (37) and using the fact that derivatives commute for a parameterized surface (37), we get

$$\begin{aligned}
\left.\frac{d^2}{dt^2}\right|_{t=0} E(F(\bullet,t)) &= -2\int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial F}{\partial s}\right\rangle ds - 2\int_a^b \left\langle \frac{\partial F}{\partial t}, R\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right)\frac{\partial F}{\partial s}\right\rangle. \\
&= -2\int_a^b \left\langle V, \frac{D^2}{\partial s^2}V\right\rangle ds - 2\int_a^b \langle V, R(V,c'), c'\rangle \, ds \\
&= -2\int_a^b \langle V, V''\rangle \, ds - 2\int_a^b \langle V, R(c',V)c'\rangle \, ds
\end{aligned}$$

We obtain the following.

Proposition 18.5 (Second variational formula)**:** eq:2nd-var-form Let (M,g) be any manifold with nondegenerate symmetric bilinear form. Let $F : [a,b]\times(-\varepsilon,\varepsilon) \to M$ be a parameterized surface that is a variation of the central curve $F(s,0) = c(s)$. Then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E = -2\int_a^b \langle V, V'' + R(c',V)c'\rangle \, ds.$$

Note the second term is the expression in the Jacobi equation! This is the second variational formula.

It is convenient to introduce the following notation for the second quantity above.

Definition 18.6: Define the **Jacobi operator** or the **second variational operator** by

$$LV := V'' + R(c',V)c'.$$

(This has nothing to do with the length defined earlier.) Using the Jacobi operator, we can rewrite the second variational formula (Proposition 18.5) as

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E = -2\int_a^b \langle V, LV\rangle \, ds.$$

Remember, all this was for a variation of curves starting and ending at the same point.

Definition 18.7: A geodesic is said to be **stable** if

$$\frac{d^2}{dt^2}E \geq 0$$

for all variations that fix the endpoints.

We've seen that a minimizing geodesic minimizes the energy. This means that for any variation of a minimizing geodesic with the same endpoints, all other competing curves will have larger energy. The second variation will be nonnegative (second derivative test); hence the geodesic is stable.

However, there can be non-minimizing geodesics for which the second derivative is negative.

Let's look at one particular example.

Example 18.8: ex:stable-geo-sphere Let $S^2 \subset \mathbb{R}^3$ be the unit sphere; it has constant curvature 1. In fact, we know that if $V \perp c'$ and $|c'| = 1$, then

$$R(c', V)c' = V.$$

Take a piece of the equator and look at the second variation.

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} E &= -2\int_a^b \langle V, V'' + V\rangle \, ds \\ &= -2\int_a^b \langle V, V''\rangle \, ds - 2\int_a^b |V|^2 \, ds \\ &= 2\int_a^b |V'|^2 \, ds - 2\int_a^b |V|^2 \, ds\end{aligned}$$

In the last line we integrated by parts, $\int \langle V, V'\rangle' = \int \langle V', V'\rangle + \int \langle V, V''\rangle$, and used that $\langle V, V'\rangle$ is 0 at a and b because V' is 0.

Since we're on a surface, we can write $V = \phi\vec{n}$. (The normal is also a parallel vector field.) Thus $V' = \phi'\vec{n}$. Requiring $V = 0$ at the beginning and end is the same as saying $\phi(a) = \phi(b) = 0$. Then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E = 2\int_a^b (\phi')^2 \, ds - 2\int_a^b \phi^2 \, ds.$$

When is this nonnegative for all ϕ with this property?

We've reduced a geometric problem to a functional inequality in calculus, called Wirtinger's inequality.

Theorem 18.9 (Wirtinger's inequality/Poincaré inequality)**:** ineq:wirtinger Let $a < b$. We have that

$$\int_a^b (\phi')^2 \geq \int_a^b \phi^2$$

for all ϕ with $\phi(a) = \phi(b) = 0$ exactly when $b - a \leq \pi$.

This means that if you take a geodesic, it is stable iff its length is at most π (so it is minimizing, a minor arc).

Proof of Theorem 18.9. If $b - a \leq \pi$ then this is a consequence of Fourier expansion. (Basically, we may assume $a = 0$; write $\phi = \sum_{n\in\mathbb{Z}} a_n e^{\frac{\pi inx}{b}}$, $\phi' = \sum_{n\in\mathbb{Z}} \frac{2\pi in}{b} a_n e^{\frac{\pi inx}{b}}$. The inequality then becomes $\sum_{n>0} \frac{\pi^2 n^2}{b^2} a_n^2 \geq \sum_{n>0} a_n^2$ for all a_n that make the sum converge. Thus $\frac{\pi^2 n^2}{b^2} \geq 1$ for all $n \geq 1$. This also motivates our choice of function below when $b - a > \pi$.)

We check the inequality fails if $b - a > \pi$. Consider

$$\phi(s) = \sin\left(\frac{s-a}{b-a}\pi\right);$$

then $\phi(a) = \phi(b) = 0$ and $\phi' = \frac{\pi}{b-a}\cos\left(\left(\frac{s-a}{b-a}\right)\pi\right)$. Then

$$(\phi')^2 = \frac{\pi^2}{(b-a)^2}\cos^2\left(\left(\frac{s-a}{b-a}\right)\pi\right).$$

If $b - a > \pi$, then the LHS is less than the RHS. We've explicitly constructed a vector field where the second derivative of energy is negative. □

Lecture 19
Thu. 11/15/12

Let (M, g) be a Riemannian manifold and let $c : [a, b] \to M$ be a curve. Recall that we defined the length $L(c) = \int_a^b |c'|\, ds$ and the energy $E(c) = \int_a^b |c'|^2\, ds$/

If we have a one parameter family of curves $F : [a, b] \times (-\varepsilon, \varepsilon) \to M$ with

$$F(s, 0) = c(s), \qquad F(a, t) = c(a), \qquad F(b, t) = c(b)$$

then we saw that

$$\frac{d}{dt}E(F(\bullet, t)) = -\int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{D}{\partial s}\frac{\partial F}{\partial s}\right\rangle ds$$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(F(\bullet, t)) = -\int_a^b \langle V, LV\rangle\, ds \qquad \text{if } c \text{ is a geodesic}$$

where $V = \frac{\partial F}{\partial t}$ and LV is defined as

$$LV = \frac{D}{\partial s}\frac{D}{\partial s}V + R(c', V)c'$$

for v a vector field along c. This is called the stability operator, second variational operator, or Jacobi operator. Note $LV = 0$ iff V is a Jacobi field.

We say a geodesic is stable if $\left.\frac{d^2}{dt^2}\right|_{t=0} E \geq 0$ for all variations that fix the endpoints.

For example, we looked at $S^2 \subseteq \mathbb{R}^3$ the unit sphere: the geodesic is stable iff it has length at most π (Example 18.8).

We have that if c is a geodesic that minimizes length, then it also minimizes energy, and hence is stable.

§1 Bonnet-Myers Theorem

We copy the argument for S^2 to get a general theorem.

Given a n-dimensional Riemannian manifold (M^n, g), the Ricci curvature is the trace of the quadratic form given by the curvature:

$$\mathrm{Ric}_M(V,V) = \mathrm{Tr}(\langle T(V,\cdot)V,\cdot\rangle)$$

where $V$is a unit vector. For instance, if $M = S^n$, then $\mathrm{Ric}_M(V,V) = n-1$.

Theorem 19.1 (Bonnet-Myers)**:** Let (M^n, g) be a n-dimensional Riemannian manifold satisfying

$$\mathrm{Ric}_M \geq (n-1)k^2$$

for some constant $k > 0$. Then M is compact, and

$$\mathrm{diam}(M) \leq \frac{\pi}{k}.$$

(Here, $\mathrm{diam}(M) = \sup_{p,q\in M} d(p,q)$.)

Bonnet proved the theorem for sectional curvature in the late 1800's; Myers generalized it to the Ricci curvature.

Proof. We can modify the metric by a constant: let

$$\widetilde{g} = k^2 g, \qquad \widetilde{M} = \left(M, k^2 g\right).$$

Then $K_{\widetilde{M}} = \frac{1}{k^2}K_M$ and it suffices to show $\mathrm{Ric}_{\widetilde{M}} \geq n-1$, i.e., it suffices to prove the statment for $k = 1$.

We use the same idea that geodesics longer than π are not stable (Example 18.8). Take two points p, q. It suffices to prove that for each pair and each minimizing geodesic γ between them, we have

$$L(\gamma) = d(p,q) \leq \pi.$$

Assume by way of contradiction that $L(\gamma) > \pi$. Suppose $\gamma : [0,\ell] \to M$ and $\ell > \pi$.

Take a parallel orthogonal frame $E_1, \ldots, E_{n-1} \in \gamma'(t)^\perp$ on γ. For each i we consider a variation that fixes the endpoints: let $V_i = \phi E_i$. where ϕ is a function such that $\phi(0) = \phi(\ell) = 0$. We look at the energy of a variation. For each i,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(V_i) \geq 0.$$

We have $LV = V'' + R(c', V)c'$ so summing these equations gives

$$0 \leq \frac{1}{2}\sum_{i=1}^{n-1} \left.\frac{d^2}{dt^2}\right|_{t=0} E(V_i) = -\int_0^\ell \langle V_i, LV_i\rangle\, ds.$$

We have $V_i = \phi E_i$ so

$$V_i' = \phi' E_i$$
$$V_i'' = \phi'' E_i.$$

Thus we get

$$\begin{aligned} 0 &\leq -(n-1)\int_0^\ell \phi''\phi - \sum_{i=1}^{n-1}\int_0^\ell \phi^2 \langle E_i, R(\gamma', E_i)\gamma'\rangle \\ &= -(n-1)\int_0^\ell \phi''\phi - \int_0^\ell \phi^2 \operatorname{Ric}(\gamma',\gamma') \\ &\leq -(n-1)\int_0^\ell \phi''\phi - (n-1)\int_0^\ell \phi^2 && \operatorname{Ric}(M) \geq n-1 \\ eq:965-19.1 \implies 0 &\geq \int_0^\ell \phi''\phi + \int_0^\ell \phi^2 && \text{for all } \phi \text{ with } \phi(0) = 0 = \phi(\ell). \end{aligned} \tag{38}$$

Taking a page from Example 18.8, we let $\phi = \sin\left(\frac{s}{\ell}\pi\right)$. Then

$$\phi' = \frac{\pi}{\ell}\cos\left(\frac{s}{\ell}\pi\right)$$
$$\phi'' = -\left(\frac{\pi}{\ell}\right)^2\cos\left(\frac{s}{\ell}\pi\right)$$

Plugging into (38) we get $\ell \leq \pi$, as needed.

Since M is bounded and complete, it must be compact. $\square$

Note the maximum possible diameter is attained by a unit sphere of radius r, so Bonnet-Myers tells us that a manifold with Ricci curvature at least c has diameter at most that of the unit sphere with curvature c.

Corollary 19.2: Suppose $(M,g) = 0$ for a Riemannian manifold (M,g) and $\operatorname{Ric}(M) \geq c > 0$. Then the fundamental group $\pi_1(M)$ is finite.

Proof. If $\widetilde{M} \to M$ is a cover, we can pull back the metric to M. Since the curvature is given by the metric, the curvature is the same on $\widetilde{M}$ and on the corresponding point on M.

Apply this to the case where $\widetilde{M}$ is the universal cover. We obtain that $\widetilde{M}$ is compact. Hence it has a finite number of sheets over M. The number of sheets equals the number of elements of $\pi_1(M)$, so $\pi_1(M)$ is finite. $\square$

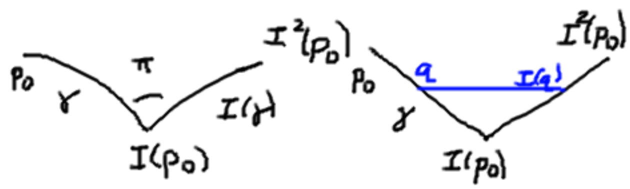
Theorem 19.3 (Synge Theorem)**:** Let M be closed (i.e., complete, compact, and without boundary) with positive sectional curvature everywhere ($K_M > 0$). Suppose the dimension of M is even, and that $I : M \to M$ is an orientation preserving isometry. Then I has a fixed point.

(If M is odd and I is an orientation reversing isometry, then the same conclusion holds. The idea is the same. See [3, Theorem 9.3.7].)

Proof. Suppose by way of contradiction that I has no fixed point. Consider the displacement function $d(p) = d_M(p, I(p))$. This is a continuous (in fact Lipschitz) map defined on a compact manifold, so it attains a minimum for some p_0:

$$\min_{p \in M} d(p) = d(p_0) > 0.$$

By completeness, we can let γ be a minimizing geodesic between p_0 and $I(p_0)$. We claim that the angle between γ and $I(\gamma)$ must be π.

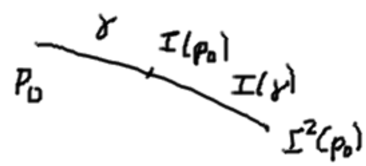


Indeed, letting q be midpoint of γ, the distance from q to $I(q)$ is at least the distance along the second half of γ and then the first half of $I(\gamma)$:

$$d(q, I(q)) \geq d(p_0) = d(q, I(p_0)) + d(I(p_0), I(q)).$$

Equality holds so the second half of γ together with the first half of $I(\gamma)$ must give a geodesic from q to $I(q)$. This means in particular that the derivative of γ and $I(\gamma)$ at $I(p_0)$ must be the same, $(I \circ \gamma)'(0) = \gamma'(\ell)$. Thus the angle between γ and $I(\gamma)$ is π.

Thus we have the following picture.



Let P be parallel translation along γ. Then we have that

$$W := dI^{-1} \circ P : T_{p_0}M \to T_{p_0}M$$

is an orientation-preserving isometry. We saw that $dI(\gamma'(0)) = \gamma'(\ell)$ above, so

$$(dI^{-1} \circ P)(\gamma'(0)) = \gamma'(0).$$

Because W is an isometry it maps the orthogonal complement to the orthogonal complement:

$$W((\gamma'(0))^{\perp}) = (\gamma'(0))^{\perp}.$$

But an isometry in Euclidean space is just made up of rotations on 2-dimensional spaces (in some basis), so there is one direction where $W(v) = v \perp \gamma'$.

Let V be the vector field along γ that is the parallel translation of v. The fact that $W(v) = v$ exactly says that the geodesic starting at $I(p_0)$ with direction $V(\ell)$ is $I(\gamma)$.

We use the second variation and $K_M > 0$ to obtain a contradiction.

Consider

$$\frac{d^2}{dt^2}E = -\int_0^\ell \langle V, LV\rangle \, ds, \qquad LV = V'' + R(\gamma', V)\gamma' = R(\gamma', V)\gamma'.$$

We have

$$\frac{d^2}{dt^2}E = -\int_0^\ell \langle V, LV\rangle \, ds = -\int_0^\ell K(\gamma', V) < 0.$$

This means that the displacement wasn't minimized at p_0 because other geodesics that are close by have shorter length, specifically, the geodesics in the variation given by V. □

Lecture 20
Tue. 11/20/12

Recall that we talked about the first and second variations of energy. Given a curve $c : [a, b] \to M$, we looked at

$$\left.\frac{d}{dt}\right|_{t=0} E(c).$$

We looked at a parameterized surfaces $F : [a, b] \times (-\varepsilon, \varepsilon) \to M$ where the central curve is a geodesic: $F(\bullet, 0) = c$. We calculated that in this case

$$\left.\frac{d}{dt}\right|_{t=0} E(c) = \left\langle \frac{\partial F}{\partial t}, c'(b) \right\rangle - \left\langle \frac{\partial F}{\partial t}, c'(a) \right\rangle.$$

In the case of fixed endpoints, this is 0. When the endpoints may vary, the formula above gives the derivative of energy.

If $F(\bullet, t)$ is a geodesic for each t, then $\frac{\partial F}{\partial t}$ is a Jacobi field, i.e., it satisfies the following second-order differential equation:

$$\frac{D}{\partial t}\frac{\partial F}{\partial t} + R\left(c', \frac{\partial F}{\partial t}\right) c' = 0.$$

Recall that solutions J to $J'' + R(c', J)c' = 0$ are uniquely given by initial data $J(a)$ and $J'(a)$.

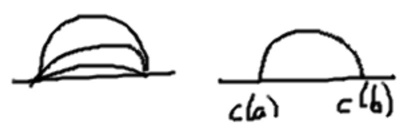
Conversely, given a geodesic c and a Jacobi field J on c, we can construct a variation such that the time derivative is J on c. Putting things together, given $v, w \in T_{c(a)}M$, we can

- construct a Jacobi field J such that $J(a) = v$ and $J'(a) = w$, and
- construct a geodesic variation so that $\frac{\partial F}{\partial t} = J$.

Every Jacobi field is infinitesimally coming from a variation of geodesics.

We had the notion of *conjugate point*: If c is a geodesic, $c(b)$ is a conjugate point for $c(a)$ along c if there exists a nontrivial Jacobi field with $J(a) = 0$ and $J(b) = 0$. There is a conjugate point for $c(a)$ if there is a variation of geodesics with the same length, starting at a and infinitesimally ending at b (i.e. are close to b with higher order).

This means that if you continue the geodesic, it can't minimize past the conjugate point: Up to higher order there is another geodesic of the same length from $c(a)$ to $c(b)$. You can move along this other geodesic to get to the further point, but it would have a corner; a minimizing path cannot have a corner. (The first variation says that you can move in and the curve will be shorter. You can easily make this rigorous.)



We can generalize from curves to surfaces or manifolds, often with weaker statements.

§1 Index form

Consider the second variation when the endpoints are fixed. We have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(F(\bullet, t)) = -\int_a^b \langle V, LV \rangle$$

where V is a vector field along c and $LV = \frac{D^2}{\partial s^2} V + R(c', V)c'$. Using $\frac{d}{ds} \langle V, V' \rangle = \langle V', V' \rangle + \langle V, V'' \rangle$ we get the above to equal

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(F(\bullet, t)) = -\int_a^b \langle V, LV \rangle = \int_a^b \langle V', V' \rangle - \langle R(c', V)c', V \rangle .$$

(Note $\langle V, V' \rangle$ is 0 at a and b.) This motivates the following definition.

Definition 20.1: Define the **index form** on c to be

$$I(V, W) := \int_a^b \langle V', W' \rangle - \langle R(c', V)c', W \rangle .$$

Note I is a symmetric bilinear form, and as a quadratic form, the index form is the second derivative of energy:

$$I(V, V) = \left.\frac{d^2}{dt^2}\right|_{t=0} E(t).$$

Let J_1, J_2 be Jacobi fields along a geodesic. Define

$$f = \langle J_1, J_2' \rangle - \langle J_1', J_2 \rangle ;$$

note this is constant. Indeed,

$$f' = \langle J_1, J_2'' \rangle - \langle J_1'', J_2 \rangle = - \langle J_1, R(c', J_2)c' \rangle + \langle R(c', J_1)c', J_2 \rangle = 0.$$

Let J_1, J_2 be Jacobi fields with $J_1(a) = J_2(a) = 0$. We then have

$$eq : 965 - 20.1 \, \langle J_1, J_2' \rangle = \langle J_1', J_2 \rangle \text{ for all } s. \tag{39}$$

This is a very useful trick: when we take the inner product of one Jacobi field with the derivative of another, we can interchange derivatives.

§2 Index lemma

Lemma 20.2 (Index lemma: Jacobi fields minimize the index form)**:** lem:index-lemma Let $c : [a, b] \to M$ be a geodesic such that there are no conjugate points to $c(a)$ along c. If J and V are vector fields along c such that J is a Jacobi field, and such that

$$J(a) = V(a) = 0$$
$$J(b) = V(b),$$

then

$$I(J, J) \le I(V, V)$$

with equality iff $J = V$.

Proof. Let $J_1, \ldots, J_{n-1}$ be Jacobi fields along c. We have $J_1(a) = \cdots = J_{n-1}(a) = 0$ and $J_1'(a), \ldots, J_{n-1}'(a)$ is an orthonormal basis for $(c'(a))^\perp$. Thus $J_1(s), \ldots, J_{n-1}(s)$ is a basis for $(c'(a))^\perp$.

For $s > 0$, we can write $V(s) = f_i(s)J_i(s)$. This is clear when $s > 0$. Since the vector field also vanishes at 0, the f_i can be extended to a smooth function including 0. This is a trivial statement about the Taylor expansion.

We claim the integrand in the index form equals

$$eq : 965 - 20.2 \, \langle V', V' \rangle - \langle R(c', V')c', V \rangle = \left\langle \sum_{i=1}^{n} f_i' J_i, \sum_{j=1}^{n} f_j' J_j \right\rangle + \frac{d}{ds} \left\langle \sum_{i=1}^{n} f_i J_i, \sum_{j=1}^{n} f_j J_j' \right\rangle. \tag{40}$$

Writing $V = f_i J_i$, we find $V' = f_i' J_i + f_i J_i'$. We have (omitting the summation sign)

$$eq : 965 - 20.3 \, \langle V', V' \rangle = f_i' f_j' \langle J_i, J_j \rangle + f_i' f_j \left\langle J_i, J_j' \right\rangle + f_i f_j' \langle J_i', J_j \rangle + f_i f_j \left\langle J_i', J_j' \right\rangle. \tag{41}$$

We have

$$\begin{aligned}
\frac{d}{ds}\left\langle f_iJ_i, f_jJ_j'\right\rangle &= \frac{d}{ds}\left(f_if_j\left\langle J_i, J_j'\right\rangle\right) \\
&= f_i'f_j\left\langle J_i, J_j'\right\rangle + f_if_j'\left\langle J_i, J_j'\right\rangle + f_if_j\left\langle J_i', J_j'\right\rangle + f_if_j\left\langle J_i, J_j''\right\rangle \\
&= f_i'f_j\left\langle J_i, J_j'\right\rangle + f_if_j'\left\langle J_i', J_j\right\rangle + f_if_j\left\langle J_i', J_j'\right\rangle - f_if_j\left\langle J_i, R(c', J_j)c'\right\rangle \quad \text{by (39)} \\
eq: 965 - 20.4 &= f_i'f_j\left\langle J_i, J_j'\right\rangle + f_if_j'\left\langle J_i', J_j\right\rangle + f_if_j\left\langle J_i', J_j'\right\rangle - \left\langle V, R(c', V)c'\right\rangle. \qquad (42)
\end{aligned}$$

From (41) and (42) we get

$$\langle V', V'\rangle - \langle R(c',V)c', V\rangle = f_i'f_j'\left\langle J_i, J_j\right\rangle + f_i'f_j\left\langle J_i, J_j'\right\rangle + f_if_j'\left\langle J_i', J_j\right\rangle + f_if_j\left\langle J_i', J_j'\right\rangle - \langle R(c',V)c', V\rangle. \quad = \frac{d}{ds}$$

which is (40). Now

$$\begin{aligned}
I(V,V) &= \int_a^b \left(\langle V', V'\rangle - \langle R(c',V)c', V\rangle\right) \\
&= \int_a^b |f_i'J_i|^2 + \left\langle f_i(b)J_i(b), f_j(b)J_j'(b)\right\rangle \\
&= \int_a^b |f_i'J_i|^2 + f_i(b)f_j(b)\left\langle J_i(b), J_j'(b)\right\rangle \\
&\geq I(J,J).
\end{aligned}$$

(Remember $J(a) = V(a) = 0$ and $J(b) = V(b) = f_i(b)J_i(b)$. Writing $J = h_iJ_i$, $h_i(b) = f_i(b)$. Note that the f_i are constants, so the first term vanishes for $I(J,J)$.) If equality holds, because $c(a)$ has no conjugate point, $f_i'(s)J_i(s) = 0$ for all s. This means $V = f_i(b)J_i$ where the f_i are constants, and we must have $V = J$. □

Next time we will prove the Rauch Comparison Theorem. We have

$$I(V,V) = \int_a^b \langle V', V'\rangle - \langle R(c',V)c', V\rangle.$$

Let $c_1 : [a,b] \to M_1$ and $c_2 : [a,b] \to M_2$. Let $E_1, \ldots, E_{n-1}$ be a parallel orthornomal parallel frame (perpendicular to the velocity vector) for $(c_1')^\perp$, and write $V = f_iE_i$. Let $\widetilde{E_i}$ be a parallel orthonormal frame on c_2 in M_2; define ϕ so that $\phi(V) = f_i\widetilde{E_i}$. We then have $\langle V(s), W(s)\rangle = \langle \phi(V(s)), \phi(W(s))\rangle$. This is a trivial but useful way of transferring vector fields between manifolds.

The index has independent interest. The index lemma shows that assuming there is no conjugate point along the geodesic, Jacobi fields minimize the index form among vector fields along the geodesic that vanish at starting point and have same value at other endpoint. Since the index form is the second derivative of energy, another way of saying this is the following.

A geodesic without conjugate points is stable: the second variation of energy is non-negative.

Lecture 21
Tue. 11/27/12

Last time we defined the index form. Let $\gamma : [a,b] \to M$ be a geodesic, and let V be a vector field along γ. Then

$$I(V,V) = \int_a^b (\langle V', V'\rangle - \langle R(\gamma', V)\gamma', V\rangle)\, ds.$$

We showed in Lemma 20.2 that if γ has no conjugate points, and V, J are vector fields along γ such that $V(a) = J(a) = 0$, $V(b) = J(b)$, and J is Jacobi then

$$I(V,V) \geq I(J,J)$$

with equality iff $V = J$.

Let $F : [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a variation such that

$$F(\bullet, 0) = \gamma, \qquad F(a, \bullet) = \gamma(a), \qquad F(b, \bullet) = \gamma(b).$$

Let $V = \frac{\partial}{\partial t} F$. Letting $LV = \frac{D^2}{\partial s^2} V + R(\gamma', V)\gamma' = V'' + R(\gamma', V)\gamma'$, we have

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0} E &= 0 \\
\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} E(F(\bullet, t)) &= -\int_a^b \langle V, LV\rangle \\
&= -\int_a^b \langle V, V''\rangle + \langle V, R(\gamma', V)\gamma'\rangle \\
&= I(V,V).
\end{aligned}$$

We used

$$\langle V, V'\rangle' = \langle V', V'\rangle + \langle V, V''\rangle$$

and noted that $\langle V, V'\rangle$ vanishes at both endpoints. (If it doesn't vanish at both endpoints, there are some additional terms.)

We'll prove the Rauch comparison theorem 21.1 and then mention what can be done in higher dimensions (see the first chapter of [2], A Course in Minimal Surfaces, AMS 2011, GTM, Colding-Minicozzi).

§1 Rauch Comparison Theorem

Suppose that M_1^n and M_2^n have the same dimension. Let γ_1, γ_2 be unit speed geodesics on M_1 and M_2, parametrized on the same interval $[a,b]$. Let $\mathfrak{X}(\gamma_i)$ be the space of smooth vector fields along γ_i. Let

$$\phi : \mathfrak{X}(\gamma_1) \to \mathfrak{X}(\gamma_2)$$

be defined as follows. Let $E_1, \dots, E_{n-1} \perp \gamma_1'$ be parallel vector fields along γ_1 and let $\widetilde{E}_1, \dots, \widetilde{E}_{n_1}$ be parallel vector fields along $\widetilde{E}_i \perp \gamma_2'$. If

$$V = f_1 E_1 + \cdots + f_{n-1} E_{n-1} + f_n \gamma_1'$$

then define

$$\phi(V) = f_1 \widetilde{E}_1 + \cdots + f_{n-1} \widetilde{E}_{n-1} + f_n \gamma_2'.$$

At any $s \in [a, b]$, we have

$$\langle V_1, V_2 \rangle (s) = \langle \phi(V_1), \phi(V_2) \rangle (s).$$

Define

$$\begin{aligned} K_1(s) &= \inf \{K(\Pi) : \Pi \text{ is a 2-plane at } \gamma_1(s) \text{ containing } \gamma_1'(s)\} \\ K_2(s) &= \sup \{K(\Pi) : \Pi \text{ is a 2-plane at } \gamma_2(s) \text{ containing } \gamma_2'(s)\}. \end{aligned}$$

Note the asymmetry. The statement is that one manifold is more curved than the other. This is almost always applied when one of the manifolds have constant curvature, in which there is no inf or sup involved.

Theorem 21.1 (Rauch comparison theorem)**:** thm:rauch Let the setup be as above. Assume there are no conjugate points along γ_i.

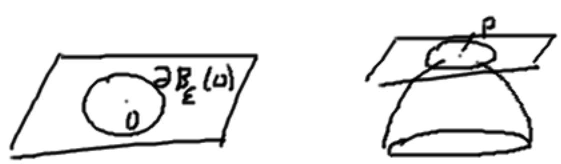
If $K_1(s) \geq K_2(s)$ for all $s \in [a, b]$, then for any pair of Jacobi fields J_1 along γ_1 and J_2 along γ_2 such that

$$J_1(a) = 0, \qquad J_2(a) = 0, \qquad |J_1'(a)| = |J_2'(a)|,$$

then

$$|J_1(b)| \leq |J_2(b)|.$$

The picture is as follows. Let $p \in M$ and consider $\exp_p : T_pM \to M$. Let $\Pi \subseteq T_pM$ be a subspace. Consider the length of $\exp(\partial B_\varepsilon(0))$. We looked at the Taylor expansion (23); the first nontrivial term is a curvature term, the sectional curvature of the 2-plane. If the sectional curvature is positive, the term has a negative sign. The length is smaller than what it is in Euclidean space.



If you take something that is positively curved, the image has smaller length than the circle that it is mapped from. Positively curved means that geodesics are spreading less rapidly than in Euclidean space. When you calculate the length of $\exp(\partial B_\varepsilon(0))$, you are calculating the derivative of the exponential map, which is given by a Jacobi field. Thus we see that Rauch Comparison 21.1 would give information about the length of $\exp(\partial B_\varepsilon(0))$ relative to $\partial B_\varepsilon(0)$.

Consider the special case where M_2 has constant sectional curvature, say 0. Say that M_2 is 2-dimensional, so M_2 is just a plane. Suppose $K_1(s)$ is positive everywhere, so J_1 is on a positively curved manifold, and J_2 is in Euclidean space. At the very beginning, these two geodesics spread out at the same rate $|J_1'(a)| = |J_2'(a)|$. The statement is that the Jacobi field in the positively curved manifold is spreading out less rapidly, $|J_1(b)| \leq |J_2(b)|$.

Proof of Theorem 21.1. Let $v_i = |J_i|^2$. Note that

$$eq:965-21-2 \frac{d}{ds}\left(\frac{v_2}{v_1}\right) \geq 0 \iff \frac{|J_2|^2}{|J_1|^2} \text{ increases.} \tag{43}$$

If we can additionally show then

$$eq:965-21-1 \lim_{s\to 0^+} \frac{|J_2|^2}{|J_1|^2} = 1, \tag{44}$$

then we get

$$|J_2(b)|^2 \geq |J_1(b)|^2 \implies |J_2(b)| \geq |J_1(b)|,$$

which is exactly what we wanted to prove.

First we show (44). Recall the Taylor expansion (22) of $h_i = |J_i|^2$. The first nontrivial term is the curvature term, which is negligible. The only term that matters is the nonzero term $|J_i'(a)|$. Thus the ratio goes to 1.

It suffices to show (43). We have

$$eq:965-21-3 0 \leq \left(\frac{v_2}{v_1}\right)' = \frac{v_2'v_1 - v_1'v_2}{v_1^2} \iff v_2'v_1 \geq v_1'v_2 \iff \frac{v_2'}{v_2} \geq \frac{v_1}{v_1'}. \tag{45}$$

(We can take the quotient because we assumed there are no conjugate points.)

We want to show that $\frac{v_2'(s_0)}{v_2(s_0)} \geq \frac{v_1'(s_0)}{v_1(s_0)}$. Define

$$U_i = \frac{J_i}{|J_i(s_0)|}, \qquad i = 1, 2.$$

Consider the index form $I(U_i, U_i)$ on $\gamma_i|_{[a,s_0]}$. By definition,

$$I(U_i, U_i) = \int_a^{s_0} (\langle U_i', U_i'\rangle - \langle R(\gamma_i', U_i)\gamma_i', U_i\rangle)\, ds.$$

If J is a Jacobi field along a geodesic $\gamma : [a, s_0] \to M$, then by definition

$$I(J, J) = \int_a^{s_0} (\langle J', J'\rangle - \langle R(\gamma', J)\gamma', J\rangle)\, ds$$

Using $\langle J', J\rangle' = \langle J'', J\rangle + \langle J', J'\rangle = \langle J', J'\rangle - \langle R(\gamma', J)\gamma', J\rangle$, get

$$I(J, J) = \langle J', J\rangle \big|_a^{s_0} = \langle J'(s_0), J(s_0)\rangle .$$

Then

$$\begin{aligned}
I(U_i, U_i) &= \langle U_i', U_i \rangle (s_0) \\
&= \left\langle \frac{J_i'(s_0)}{|J_i(s_0)|}, \frac{J_i(s_0)}{|J_i(s_0)|} \right\rangle \\
&= \frac{\langle J_i'(s_0), J_i(s_0) \rangle}{|J_i(s_0)|^2} \\
&= \frac{1}{2} \frac{v_i'(s_0)}{v_i(s_0)}.
\end{aligned}$$

Thus (44) is equivalent to $I(U_2, U_2) \geq I(U_1, U_1)$. We have to prove an inequality about index forms, so it's helpful to have a way to move vector fields between γ_1 and γ_2. This is where ϕ comes in!

(ϕ was almost canonical but involved a choice of parallel frame. We identify an orthonormal basis at one point.)

By choice of orthonormal basis $\widetilde{E}_i$ we may assume $\phi(U_2)(s_0) = U_1(s_0)$ (the tangential components have to be the same). Now

$$\begin{aligned}
I(U_2, U_2) &= \int_a^{s_0} (\langle U_2', U_2' \rangle - \langle R_{M_1}(\gamma_2', U_2)\gamma_2', U_2 \rangle)\, ds \\
&= \int_a^{s_0} (\langle U_2', U_2' \rangle - |U_2|^2 K(\gamma_2', U_2))\, ds
\end{aligned}$$

Now note if $V = f_1 E_1 + \cdots + f_{n-1} E_{n-1} + f_n \gamma'$, then $\phi(V) = f_1 \widetilde{E_1} + \cdots + f_{n-1} \widetilde{E}_{n-1} + f_n \gamma'$. We have $V' = f_1' E_1 + \cdots + f_{n-1}' E_{n-1} + f_n \gamma'$ and $\phi(V') = f_1' \widetilde{E_1} + \cdots + f_{n-1}' \widetilde{E}_{n-1} + f_n' \gamma'$. Then (note $\phi(U_2)$ may not be Jacobi)

$$\begin{aligned}
I(\phi(U_2), \phi(U_2)) &= \int_a^{s_0} (\langle \phi(U_2)', \phi(U_2)' \rangle - \langle R_{M_1}(\gamma_2', \phi(U_2))\gamma_2', \phi(U_2) \rangle)\, ds. \\
&= \int_a^{s_0} (\langle \phi(U_2)', \phi(U_2)' \rangle - |\phi(U_2)|^2 K(\gamma_2', \phi(U_2)))\, ds
\end{aligned}$$

Thus using the fact that M_1 is more curved than M_2, we get

$$I(U_2, U_2) \geq I(\phi(U_2), \phi(U_2)).$$

But $\phi(U_2)$ is a vector field along $\gamma_1|_{[a,s_0]}$. At 0 it is 0 and at s_0, we arranged for $\phi(U_2)(s_0) = U_1(s_0)$. Since U_1 is a Jacobi field with same vector value, by the minimizing property (since there is no conjugate point), we get

$$I(U_2, U_2) \geq I(\phi(U_2), \phi(U_2)) \geq I(U_1, U_1).$$

This is exactly what we wanted to prove. □

Note that we actually proved something a bit stronger, that the ratio $\frac{|J_2|}{|J_1|}$ is nondecreasing.

Example 21.2: Consider manifolds M_1^n, M_2^n. Suppose M_2 has constant sectional curvature, $K_{M_2} = c^2$, 0, or $-c^2$. Consider Jacobi fields that initially vanish. Let $J = J_2$.

- If $K_{M_2} = c^2$, $J(0) = 0$, $|J'(0)| = 1$, and J is orthogonal to the geodesic, then we can write
$$J = \frac{1}{c}\sin(cs)E$$
where E is a parallel vector field. The statement is then
$$|J_1(0)| \le \frac{|J_1'(0)|}{c}\sin(cs) = |J_2|.$$
Thus the Jacobi field must vanish no later than $\frac{\pi}{c}$. Rauch comparison holds up to that point. This is a standard way that Rauch comparison is applied. (The other is when M_1 has constant sectional curvature, and the inequalities are reversed.)

- If $K_{M_2} = 0$, we can write
$$J(s) = sE.$$

- If $K_{M_2} = -c^2$, then we can write
$$J = \frac{1}{c}\sinh(cs)E.$$

Lecture 22
Thu. 11/29/12

§1 Comparing volumes

Last time we showed that if we have two Riemannian manifold (M_1^n, g) and (M_2^n, g), and
$$\sup_{\gamma_1} K_{M_1} \le \inf_{\gamma_2} K_{M_2}$$
where $\gamma_i : [0, \ell] \to M_i$ are a unit speed geodesic with no conjugate points, then we have the Rauch Comparison Theorem 21.1: If J_i are Jacobi fields along γ_i with $J_i(0) = 0$, $|J_1'(0)| = |J_2'(0)|$, then
$$\frac{d}{ds}\left(\frac{J_1}{J_2}\right) \ge 0.$$

As a consequence, we have the following.

Corollary 22.1: Assume the conditions above, and that the manifolds are complete.
If M_1 has constant curvature, $K_{M_1} = c$, and $K_{M_2} \ge c = K_{M_1}$, then
$$\mathrm{Vol}(B_r^{M_1}(p_1)) \ge \mathrm{Vol}(B_r^{M_2}(p_2)).$$

Proof. Let $M = M_2$. Consider the exponential map $\exp_p : T_pM \to M$.

We claim that $B_r(p)$ is the image of $B_r(0) \subset T_pM$ under $\exp_p$:

$$B_r(p) = \exp_p(B_r(0)).$$

Because the manifold is complete, for any $q \in B_r(p)$ there exists a minimizing unit speed geodesic from p to q. The length of γ is $d(p,q) \leq r$. Hence $B_r(p) \subseteq \exp_p(B_r(0))$. The other inclusion is clear, because the a point in the image of $B_r(0)$ is connected to p by a geodesic of length less than r. We have to be careful, however, about overcovering.

Definition 22.2: Define the **cut locus** Cut_p in T_pM as the set of points $y \in T_pM$ such that the map $s \mapsto \exp_p(sy)$ for $0 \leq s \leq 1$ is a minimizing geodesic.

Note these geodesics have no conjugate points. Observe that Cut_p is star convex: if $y \in \mathrm{Cut}_p$ then the line segment joining 0 and y is in Cut_p.

We see that

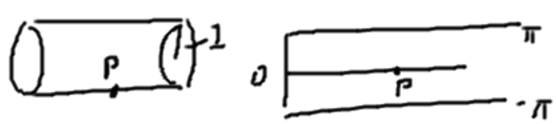
$$B_r(p) = \exp_p(\mathrm{Cut}_p \cap B_r(0)).$$

Note that $\mathrm{Cut}_p \cap B_r(0) \to B_r(p)$ is now a diffeomorphism; there is no overcovering.

As an example, take the round unit sphere. We have

$$\mathrm{Cut}_p = \overline{B_\pi(0)} \subseteq T_pM.$$

Note that $\partial B_\pi(0)$ is mapped to a single point, but the boundary doesn't contribute to the volume. As another example, consider the cylinder of radius 1.



Then the cut locus is given by $\mathbb{R} \times [-\pi, \pi]$.

Suppose first for simplicity M is a manifold with $K_M \geq 0$ and $M_1 = \mathbb{R}^n$. We need to show that $\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_r(0))$ where $B_r(0) \subseteq \mathbb{R}^n$. We have $\mathrm{Cut}_p \cap B_r(0) \to M$. Note that Jacobi fields along the geodesics given by segments in $\mathrm{Cut}_p \cap B_r(0)$ have no conjugate points, because they are inside the image of the cut locus. Geodesics have to minimize, so there cannot be conjugate points.

By the Rauch Comparison Theorem 21.1, $\frac{|J^{\mathbb{R}^n}|}{|J|}$ is increasing. We have $|J| \leq |J^{\mathbb{R}^n}|$. This says that the derivative is less than or equal to the derivative Euclidean space, where it is the identity. From this we can get the inequality for volumes.

In general, to show $\mathrm{Vol}(B_r^{M_1}(p_1)) \geq \mathrm{Vol}(B_r^{M_2}(p_2))$, consider the map

$$B_r^{M_1}(p_1) \xrightarrow{\exp_p^{-1}} T_{p_1}M_1 \xrightarrow{I} T_{p_2}M_2 \xrightarrow{\exp_p} B_r^{M_2}(p_2).$$

Calculating the Jacobian of this map and using the Rauch Comparison Theorem as before gives the inequality. (In the case one of the M_i is Euclidean space, the exponential map is the identity, and we reduce to the first case.) □

§2 Matrix Ricotti equation

Let M be a Riemannian manifold and γ be a unit speed geodesic. We defined the second variational operator $LV = V'' + R(\gamma', V)\gamma'$ where V is a vector field along γ. The Jacobi equation is $LV = 0$; any V satisfying this is a Jacobi field.

Consider a Jacobi field that vanishes initially, $J(0) = 0$. Let $E_1, \dots, E_{n-1}$ be an orthonormal parallel vector fields along γ, all orthogonal to γ'. For $J \perp \gamma$, we can write $J = j_1 E_1 + \cdots + j_{n-1} E_{n-1}$.

Consider $n-1$ linearly independent Jacobi fields with $J_i(0) = 0$ and $J_i'(0) = E_i(0)$. Then any Jacobi field $J \perp \gamma$ satisfying $J(0) = 0$ can be written $J = c_1 J_1 + \cdots + c_{n-1} J_{n-1}$.

Define a matrix $A = (a_{ij})$ to be a $(n-1) \times (n-1)$ matrix-valued function along $[0, \ell]$, where the jth column are the coefficients in the linear combination

$$J_j = \sum_{i=1}^{n-1} a_{ij} E_i.$$

By definition $A' = (a'_{ij})$ and $A'' = (a''_{ij})$. Now $J_j' = \sum_{i=1}^{n-1} a'_{ij} E_i$ and $J_j'' = \sum_{i=1}^{n-1} a''_{ij} E_i$. The Jacobi equation is

$$J_j'' + R(\gamma', J_j)\gamma' = 0.$$

We would like the Jacobi equation to give a equation—some ODE—for the matrix A. (Everything we do with matrices can be found in [1].)

Now $R(\gamma'(s), \bullet)\gamma'(s)$ is a symmetric map $T_{\gamma(s)}M \to T_{\gamma(s)}M$. We can think of this as a map $(\gamma'(s))^\perp \to (\gamma'(s))^\perp$. Let $R = (R_{ij})_{1 \le i,j \le n-1}$ be the matrix representing this operator in the basis E_i. Now $A'' = (J_1'', \dots, J_{n-1}'')$. The Jacobi equation $J'' + R(\gamma', J)\gamma' = 0$ now becomes

$$A'' + RA = 0. \tag{46}$$

Indeed, this is just $a''_{ij} + R_{ik} a_{kj} = 0$. The advantage of this equation is that you can think of A, R as functions $[0, \ell] \to \mathcal{M}_{(n-1)\times(n-1)}(\mathbb{R})$. Note both R, A are symmetric. Why are we interested in writing the Jacobi equation like this? If A is invertible, consider

$$U = A'A^{-1}.$$

(Note that $A(0) = 0$ but $\frac{d}{dt}A(0) = I$ so A is invertible for small t; it is invertible as long as there is no conjugate points. If A does not have full rank, then there is a nontrivial linear combination of Jacobi fields at that point, i.e., there is a conjugate point.) Using $(AB)' = A'B + AB'$, we hve

$$0 = I' = (AA^{-1})' = A'A^{-1} + A(A^{-1})' \implies (A^{-1})' = -A^{-1}A'A^{-1}.$$

Now

$$\begin{aligned} U' &= A''A^{-1} + A'(A^{-1})' \\ &= A''A^{-1} + A'(-A^{-1}A'A^{-1}) \\ &= A''A^{-1} - (A'A^{-1})^2 \\ &= -(RA)A^{-1} - U^2 = -R - U^2. \end{aligned}$$

We get

$$\textcolor{red}{eq:ricotti} U' + U^2 + R = 0. \tag{47}$$

This is called the *Matrix Ricotti equation*. The advantage of this equation is that it is a first order equation; the disadvantage is that it is not a linear equation.

The second reason A is so useful is that $\det(A)$ is the Jacobian of $\exp_{\gamma(s)}$.

> The Ricotti equation is useful for getting bounds on areas and volumes.

At each point of the geodesic we can take the trace of (47) to get

$$\textcolor{red}{eq:ricotti-trace} \mathrm{Tr}(U)' + \mathrm{Tr}(U^2) + \mathrm{Tr}(R) = 0. \tag{48}$$

But

$$\mathrm{Tr}(R) = \sum_{i=1}^{n-1} R_{ii} = \sum_{i=1}^{n-1} \langle R(\gamma', E_i)\gamma', E_i\rangle = \mathrm{Ric}_{\gamma(s)}(\gamma'(s))$$

so we can rewrite (48) as

$$\mathrm{Tr}(U)' + \mathrm{Tr}(U^2) + \mathrm{Ric}_{\gamma(s)}(\gamma'(s)) = 0.$$

Note if B is a symmetric $(n-1)\times(n-1)$ matrix, then the Cauchy-Schwarz inequality gives

$$\mathrm{Tr}(B) \le (n-1)\mathrm{Tr}(B^2).$$

Then we obtain

$$\mathrm{Tr}(U)' + \frac{\mathrm{Tr}(U)^2}{n-1} + \mathrm{Ric}_{\gamma(s)}(\gamma'(s)) \le 0.$$

Defining $u = \mathrm{Tr}(U)$, this can be written more simply as

$$\textcolor{red}{eq:ricotti-ineq} u' + \frac{u^2}{n-1} + \mathrm{Ric}_{\gamma(s)}(\gamma'(s)) \le 0. \tag{49}$$

This is called the *Ricotti inequality*. We've eliminated the matrices by taking the trace, but now we only have inequality. This differential inequality is useful because it is easy to estimate. In particular, if $\mathrm{Ric} \ge 0$, then

$$u' + \frac{u^2}{n-1} \le 0.$$

Let v be u but in Euclidean space. Then $\mathrm{Ric} = 0$, and in the Cauchy-Schwarz inequality we have equality iff only the identity and B are constant multiples of each other. If you write down what A is on any space form it is a constant function times the identity. (On Euclidean space it's linear, on the sphere it's sine, on the hyperbolic space it's sinh.) Thus the Cauchy-Schwarz inequality is actually equality. In the two inequalities applied (Cauchy and Ric), we have equality

$$v' + \frac{v^2}{n-1} = 0.$$

If we have 2 solutions that are initially the same, a simple Ricotti comparison argument with

$$[(u-v)e^{\int(u+v)}]' = [(u^2-v^2) + (u'-v')]e^{\int(u+v)} \geq 0$$

gives a sign on the derivative. Ricotti is very useful in estimating volumes.

Lecture 23
Tue. 12/4/12

We'll start with the Gauss-Bonnet Theorem.

§1 Gauss-Bonnet Theorem

Theorem 23.1 (Gauss-Bonnet)**:** Let M^2 be a complete Riemannian manifold. Let $p \in M$, and $B_r(p)$ be a ball of radius r around p. Suppose every geodesic starting at p going out to radius r is minimizing.

Then

$$2\pi = \int_{\partial B_r(p)} k_g \, ds + \int_{B_r(p)} k \, dA$$

Proof. Let $J = j\vec{n} \perp \gamma'$ be a Jacobi field on the variation of geodesics $\exp_p((s,\theta))$ ((s,θ) polar coordinates), satisfying $j'' + kj = 0$.

Integrating the Jacobi equation gives

$$\begin{aligned}
0 &= \int_0^r \int_0^{2\pi} (j'' + kj) \, d\theta ds \\
&= \int_0^r \int_0^{2\pi} j'' \, d\theta ds + \int_0^r \int_0^{2\pi} kj \, d\theta ds \\
&= \int_0^{2\pi} [j'(r) - j'(0)] \, d\theta + \int_{B_r(p)} k \, dA \\
&= \int_0^{2\pi} j'(r) \, d\theta - 2\pi + \int_{B_r(p)} k \, dA \\
&= \int_0^{2\pi} \frac{j'(r)}{j(r)} j(r) \, d\theta - 2\pi + \int_{B_r(p)} k \, dA
\end{aligned}$$

Here we use the fact that the geodesic is minimizing, so nothing is overcovered. We write it as above because $j(r)$ is the length element, and $\frac{j'(r)}{j(r)}$ is the geodesic curvature of $\partial B_r(p)$. Indeed, we have $\langle \nabla_e \vec{n}, e \rangle = \frac{j'(r)}{j(r)}$ where $|e|$ is tangent to $\partial B_r(p)$ (why?). (Recall that taking $N^{n-1} \subseteq M^n$, the second fundamental form, for X tangent to N, satisfies

$$\langle \nabla_X X, n \rangle = - \langle \nabla_X n, X \rangle .$$

The geodesic curvature of boundary is just the principal curvature—for a curve, there is only one principal curvature because it is one-dimensional.)

We get that

$$2\pi = \int_{\partial B_r(p)} k_g \, ds + \int_{B_r(p)} k \, dA$$

as long as all geodesics starting from p going to radius r are minimizing. $\square$

We need the following simple fact. If $\gamma : [0, r] \to M$ is a geodesic on (M^n, g), and we take a variation of γ, $F : [0, r] \times (-\varepsilon, \varepsilon) \to M$ with $F(\bullet, 0) = \gamma$, $F(0, t) = \gamma(0)$, $F(r, t) = \gamma(r)$, we found

$$\frac{d^2 E}{dt^2} = -\int \langle V, LV \rangle$$

where $V = \frac{\partial F}{\partial t}\Big|_{t=0}$ and $LV = \frac{D^2}{\partial s^2} V + R(\gamma', V)\gamma'$. If M^2 were a surface, and $V \perp \gamma'$ then we could write $V = \phi\vec{n}$ and $L\phi = \phi'' + k\phi$.

Now Let $u : \Omega \to \mathbb{R}$ where $\Omega \subseteq \mathbb{R}^n$. We consider the **Schrödinger operator**

$$Lu = \Delta u + ku.$$

Recall that we call γ stable if

$$\frac{d^2}{dt^2}\Big|_{t=0} E \geq 0$$

for all variations fixing the endpoints. We saw that by Cauchy-Schwarz that if γ minimizes length, then it is stable. If we have a surface, to say that γ is stable is the same as saying that $-\int \phi L\phi \geq 0$ for all ϕ with compact support. This is equivalent to $\int (\phi')^2 \geq \int k\phi^2$.

Suppose $u > 0$ and $Lu = 0$. We claim that $-L \geq 0$. Consider $v = \ln u$. We have

$$\Delta v = ((\ln u)')' = \left(\frac{u'}{u}\right)' = \frac{u''}{u} - \left(\frac{u'}{u}\right)^2 = \frac{u''}{u} - (v')^2$$

so

$$\Delta v = k - |\nabla v|^2.$$

For ϕ with compact support, we have by integration by parts (there is no boundary term),

$$\int \phi L\phi = -\int \phi(\Delta\phi + k\phi) = \int |\nabla\phi|^2 - \int k\phi^2,$$

and this is ≥ 0 iff

$$\int |\nabla\phi|^2 \geq \int k\phi^2.$$

Also by integration by parts we have

$$\int \phi^2 \Delta v = 2\int \phi\nabla\phi\nabla v.$$

We use the inequality $2ab \le a^2 + b^2$ (which comes form $(a+b)^2 \ge 0$). Letting $a = \phi|\nabla v|$ and $b = |\nabla v|$ we get

$$\begin{aligned}\left|\int \phi^2 \Delta v\right| &= \left|2\int \phi \nabla\phi \nabla v\right| \\ &\le 2\int |\phi||\nabla\phi||\nabla v| \\ &\le \int \phi^2|\nabla v|^2 + \int |\nabla\phi|^2.\end{aligned}$$

On the other hand, using the calculation for Δv and $0 = Lu = \Delta u + ku$, i.e., $\Delta u = -ku$, we get

$$\Delta v = -k - |\nabla v|^2.$$

We get

$$\int \phi^2\Delta v = -\int \phi^2 k - \int \phi^2|\nabla v|^2$$

which becomes

$$-\int \phi^2 k - \int \phi^2|\nabla v|^2 = \int \phi^2 \Delta v = -\left|\int \phi^2\Delta v\right| \ge -\int \phi^2|\nabla v|^2 - \int |\nabla\phi|^2.$$

We get exactly $-L \ge 0$.

We could make the same computation on any manifold M and $\Omega \subseteq M$, using the Laplacian on M.

On M^2 let γ be a geodesic, not necessarily minimizing, and let J be a Jacobi field. Write $J = j\vec{n}$. For $|J| > 0$ we have $j'' + kj = 0$, giving γ is stable.

§2 Higher dimensions

Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a surface. We want to generalize geodesics to higher dimensions. Instead of looking at the energy, we look at the area.

Let $F : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$, with $F(x, 0) = x$. Suppose that

$$F(\bullet, t)|_{\partial\Sigma} = \mathrm{id}_\Sigma,$$

so $F(x,t) = x$ if $x \in \partial\Sigma$.

Let n be the unit normal to Σ. Recall that we defined $H = \mathrm{div}_\Sigma(\vec{n})$. We have $H \equiv 0$ iff Σ is a minimal surface.

Now (I don't really get this)

$$\frac{d}{dt}\mathrm{Area}(F(\Sigma, t)) = \int_\Sigma \langle V, H\vec{n}\rangle$$

where $V := \left.\frac{\partial F}{\partial t}\right|_{t=0}$. The second variation of area is, assuming $V = \phi\vec{n}$,

$$\frac{d^2}{dt^2}\mathrm{Area}(F(\Sigma, t)) = -\int \phi L\phi$$

where

$$L = \Delta_\Sigma \phi + |A|^2 \phi$$

and A is the second fundamental form. We have

$$|A|^2 = \kappa_1^2 + \kappa_2^2,$$

for κ_1, κ_2 the principal curvatures.

Note $\kappa_1 + \kappa_2 = H = 0$ iff $\kappa_1 = -\kappa_2$. Since $K = \kappa_1 \kappa_2$, we get

$$|A|^2 = \kappa_1^2 + \kappa_2^2 = 2\kappa_1\kappa_2 = -2K.$$

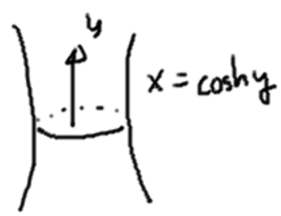
For a minimal surface the Schrödinger (Jacobi) operator is

$$L = \Delta_\Sigma \phi + |A|^2 \phi = \Delta_\Sigma \phi - 2k\phi.$$

A minimal surface is said to be stable if $0 \leq -\int_\Sigma \phi L \phi$.

Example 23.2: $\mathbb{R}^2 \subseteq \mathbb{R}^3$ is a minimal surface.

Example 23.3: Rotating $x = \cosh y$ around the y-axis, an easy computation (by Euler, 1740) shows that this is a minimal surface. It is called a **catenoid**.



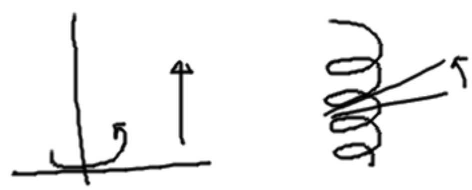
Take a sphere, if rescale everything by same factor. Any rescaling of a minimal surface is a minimal surface. If you rescale the neck it's still a minimal surface. This gives a 1-parameter family of minimal surfaces. If you translate, it is still a minimal surface.

There are catenoids with arbitarily small necks.

Catenoids are not stable. However, they have finite instability: there are a finite number of directions where the operator is negative.

We will see later that the catenoid is not stable. However, a region that doesn't contain too much of the neck is stable.

Example 23.4: (from 1776) Consider the helix $(r\cos t, r\sin t, t)$. It makes a minimal surface called a **helicoid**. You can form a helicoid by rotating a line and moving upwards at constant speed. If you rotate half of the line, you get a single spiral staircase; half of the helicoid is stable.



For the helicoid, every time you complete a rotation, you get instability. The helicoid has infinite instability.

Let $\Sigma \subseteq \mathbb{R}^3$ be a minimal surface. Let $p \in \Sigma$ and consider $\mathcal{B}_r(p)$; suppose it does not intersect $\partial\Sigma$. Suppose Σ is stable. We would like an area bound for $\mathcal{B}_r(p)$.

Considering the helicoid, note that a dilation can make the line rotating very fast as you move up. As the line rotates faster and faster, it sweeps out a larger and larger area in Euclidean space.

One way to go between two points on the helicoid is going into the axis, and down the "flight of stairs." Thus distance between any two points in an Euclidean ball is finite no matter how many times you rotate around. A general minimal surface like the helicoid; there is no area bound. We have an area bound if the surface is stable.

We have (remember $Lu = \Delta u + |A|^2 = \Delta u - 2ku$)

$$0 \leq -\int_\Sigma \phi L\phi \iff \int |\nabla\phi|^2 \geq -2\int ku.$$

Let

$$\phi = \begin{cases} 1 - \frac{d_\Sigma(p,\bullet)}{r} & \text{on } \mathcal{B}_r(p), \\ 0, & \text{otherwise.} \end{cases}$$

(this may not be smooth, but assume that in $B_r(p)$ all geodesics minimize). Suppose $|\nabla\phi| = \frac{1}{r}$ on $\mathcal{B}_r(p)$.

Using $\int |\nabla\phi|^2 \geq -2\int ku$ we get

$$\begin{aligned} \frac{\operatorname{Area}(\mathcal{B}_r(p))}{r^2} = \frac{1}{r^2}\int_{\mathcal{B}_r(p)} 1 &\geq -2\int_{\mathcal{B}_r(p)} k\left(1 - \frac{d}{r}\right)^2 \\ &= -2\int_0^r \int_{\partial\mathcal{B}_s(p)} k\left(1-\frac{d}{r}\right)^2 \\ &= -2\int_0^r \left(\left(1-\frac{s}{r}\right)^2 \int_{\partial\mathcal{B}_s(p)} k\right) ds \end{aligned}$$

We integrate by parts so we can use the Gauss-Bonnet Theorem. We have $\int_{\partial B_r(p)} k_g = \int_0^{2\pi} j'(r)\, d\theta = \left(\int_0^{2\pi}\right)'(r)$. Let $\ell(r)$ be the length of $\partial\mathcal{B}_r(p)$. It is just the integral $\ell(r) = \int_0^{2\pi}$. Hence we see

$$\ell'(r) = \int_{\partial B_r(p)} k_g.$$

Hence we can write the Gauss-Bonnet Theorem as

$$0 = \ell'(r) + \int_{B_r(p)} k = \ell'(r) + \int_0^r \int_{\partial \mathcal{B}_s(p)} k.$$

We have

$$\ell' = 2\pi - \int_{\mathcal{B}_r(p)} k.$$

By the Fundamental Theorem of Calculus, $\ell'' = -\int_{\partial B_s} k$. Continuing our calculation, integrating by parts gives

$$\begin{aligned}
\frac{\text{Area}(\mathcal{B}_r(p))}{r^2} &= -2\int_0^r \left(\left(1-\frac{s}{r}\right)^2 \int_{\partial\mathcal{B}_s(p)} k\right) ds \\
&= 2\int_0^r \left(1-\frac{s}{r}\right)^2 \ell''\, ds \\
&= \left[\left(1-\frac{s}{r}\right)^2 \ell'\right]_0^r - \frac{2}{r}\int_0^r \left(1-\frac{s}{r}\right)\ell' \\
&= 0 - \ell'(0) - \frac{2}{r}\int_0^r \left(1-\frac{s}{r}\right)\ell' \\
&= -2\pi - \frac{2}{r}\int_0^r \left(1-\frac{s}{r}\right)\ell' \\
&= -2\pi - \underbrace{\frac{2}{r}\left[\left(1-\frac{s}{r}\right)\ell\right]_0^r}_{0} + \frac{2}{r}\left(-\frac{1}{r}\right)\int_0^r \ell \\
&= -2\pi - \frac{2}{r^2}\int_0^r \ell \\
&= -2\pi - \frac{2}{r^2}\text{Area}(\mathcal{B}_r(p)).
\end{aligned}$$

We have $\int |\nabla\phi|^2 \ge -2\int k\phi^2$. For our choice of ϕ,

$$\frac{\text{Area}(\mathcal{B}_r(p))}{r^2} \ge -4\pi + \frac{4\text{Area}(\mathcal{B}_r(p))}{r^2}.$$

We get

$$\frac{4}{3}\pi \ge \frac{\text{Area}(\mathcal{B}_r(p))}{r^2}.$$

Thus we get a bound for the area of a ball.

Lecture 24
Thu. 12/6/12

Today is the last day of class.

Last time we looked at a minimal surface $\Sigma \in \mathbb{R}^3$. We assumed that Σ was stable, which meant that for all ϕ with compact support on Σ, the variation of area is nonnegative:

$$-\int_\Sigma \phi L\phi \geq 0.$$

Here L is the Laplacian,

$$L\phi = \Delta_\Sigma \phi + |A|^2\phi,$$

and $|A|^2 = \kappa_1^2 + \kappa_2^2 = -2K$ where κ_1, κ_2 are the principal curvatures.

Last time we proved, using the Gauss-Bonnet Theorem, that for $p \in \Sigma$,

$$\mathrm{Area}(\mathcal{B}_r(p)) \leq \frac{4}{3}\pi r^2.$$

When we did this calculation, we assumed there are no cut points, so $\exp_p : B_r(0) \to \mathcal{B}_r(p)$, $B_r(0) \subseteq T_pM$, is a diffeomorphism.

When you have an operator

$$L\phi = \Delta_\Sigma \phi + V\phi$$

for some "potential" V, the eigenfunctions of L are those such that

$$L\phi + \lambda\phi = 0$$

for some constant λ. We say ϕ has eigenvalue λ. (Note the sign convention.) If Σ is compact then one can prove L is a compact operator, so there is a basis of eigenfunctions. We can order the eigenfunctions ϕ_i where the associated eigenvalues satisfy

$$\lambda_1 \leq \cdots \leq \lambda_i \leq \cdots, \qquad \lambda_i \to \infty.$$

All we need to know is that if we take the eigenfunction ϕ with lowest eigenvalue λ, then ϕ cannot change sign:

$$|\phi| > 0.$$

This is easy to prove; we'll come back to it. By replacing ϕ by $|\phi|$ we may assume ϕ is positive.

If Σ is stable, then

$$-\int_\Sigma \phi L\phi \geq 0$$

where $L\phi + \lambda_1\phi = 0$, $\phi > 0$. We get

$$0 \leq -\int_\Sigma \phi L\phi = \lambda \int_\Sigma \phi^2;$$

the lowest eigenvalue is nonnegative. We obtain

$$L\phi = -\lambda\phi \leq 0.$$

If we have a Schrödinger operator $\Delta + V$, and $\phi > 0$ that is a solution to $(\Delta + V)\phi = 0$, then for all ϕ with compact support,

$$\int \psi(\Delta + V)\psi \geq 0.$$

(We looked at $\ln \phi$.) It doesn't need to be a solution, it just needs to be a *supersolution*, i.e., satisfy $L\phi + \lambda\phi \leq 0$. At some point we used an absorbing inequality; it still holds.

Now if we have $p \in \Sigma \subseteq \mathbb{R}^3$ with Σ stable, $\kappa_1 = -\kappa_2$, and $K \leq 0$, then Hadamard's Theorem gives that $\exp_p : T_p\Sigma \to \Sigma$ is a covering map. (Here we are assuming Σ is complete noncompact.) Note that in our inequality we assumed there were no cut points. We have $\Sigma \subseteq \mathbb{R}^3$; pulling back the metric we get a covering map (locally an isometry). The composition

$$T_p\Sigma \xrightarrow{\exp_p} \Sigma \subseteq \mathbb{R}^3$$

is an immersion. But in $T_p\Sigma$ the exponential map is actually a diffeomorphism; there is no cut point. By going to the cover, we can assume there is no cut point. We don't have the assumption that T_pM is stable, but we show this is true. Consider the operator $L = \Delta_\Sigma + |A_\Sigma|^2$. Take the eigenfunction corresponding to the smallest eigenvalue, we may assume $\phi > 0$. Then $L_\Sigma\phi \leq 0$. Composing with the exponential map, we may consider it on T_pM:

$$\widetilde{\phi} = \phi \circ \exp_p.$$

Let $\widetilde{\Sigma} = T_pM$ be T_pM with the pullback metric. We have

$$L_{\widetilde{\Sigma}}\phi \leq 0.$$

This implies $\widetilde{\Sigma}$ is stable. We have an inequality for all functions. On the cover there are many more functions than pullback functions, so we have to prove something more.

We've removed the cut point assumption; we always have the inequality on area; we have the inequality for $\widetilde{\Sigma}$, so we clearly also have it on Σ. The area of the corresponding ball on Σ is smaller than the pullback area.

§1 Curvature estimate

We're aim to prove a curvature estimate.

Theorem 24.1: If Σ is stable, then if $\mathcal{B}_r(p) \subseteq \Sigma \backslash \partial\Sigma$, then

$$\sup_{\mathcal{B}_{\frac{r}{2}}} |A|^2 \leq Cr^{-2}$$

where C is a constant independent of r and p.

If we have a minimal surface, its image under any isometry is a minimal surface. In $\mathbb{R}^3$, any scaling of a minimal surface is a minimal surface. Thus we obtain a whole family of minimal surfaces. For instance, we can make the neck of a catenoid as small as we want. The first surface is stable iff the image is. (Actually the index is the same.)

A catenoid cannot be stable. Take a huge ball (let $r \to \infty$); the half ball has almost 0 curvature. But in the middle there's curvature. Thus a catenoid is not stable. The same argument works for the helicoid.

To prove our theorem, we need the following.

1.1 Logarithmic cut-off trick

Let Σ be a minimal surface, and suppose we have a quadratic area bound

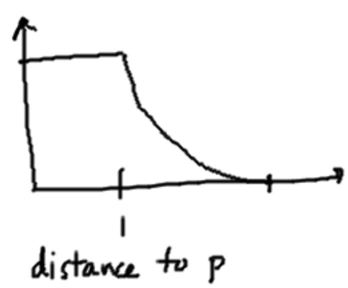
$$\operatorname{Area}(\mathcal{B}_r(p)) \le cr^2.$$

We show that if this holds for all r, then we can find a function that is 1 on the unit ball centered at the point, and has small energy.

Define

$$\phi = \begin{cases} 1 & \text{on } \mathcal{B}_1(p) \\ 1 - \frac{\ln s}{\ln r} & \text{on } \partial\mathcal{B}_s(p),\ 1 \le s \le r \\ 0 & \text{otherwise.} \end{cases}$$

The function decays from 1 to 0 from 1 to r.



We calculate that

$$|\nabla\phi| = \begin{cases} 0 & \text{on } \mathcal{B}_1(p) \cup (\Sigma \backslash \mathcal{B}_r(p)) \\ \frac{1}{s \ln r} & \text{on } \partial\mathcal{B}_r(p),\ 1 < s < r. \end{cases}$$

Note ϕ has compact support because it dies at r. We show that if r is large, ϕ has small energy:

$$\begin{aligned} \int |\nabla\phi|^2 &= \int_0^\infty \int_{\partial\mathcal{B}_r(p)} |\nabla\phi|^2 \, d\ell ds \\ &= \int_1^r \left(\frac{1}{s \ln n}\right)^2 d\ell ds \\ &= \frac{1}{(\ln r)^2} \int_1^r \frac{\ell(s)}{s^2} \, ds. \end{aligned}$$

(Integrate over the distance spheres.) Using

$$\text{Area}(\mathcal{B}_r(p)) = \int_0^r \ell(s)\, ds$$

we have

$$\frac{d}{ds}\text{Area}(\mathcal{B}_r(p)) = \ell(r).$$

We integrate by parts because we don't have a bound for ℓ but we have a bound for area. We get

$$\begin{aligned}
\int |\nabla\phi|^2 &= \frac{1}{(\ln r)^2}\int_1^r \frac{\ell(s)}{s^2}\, ds \\
&= \frac{1}{(\ln r)^2}\left(\left[\frac{\text{Area}(\mathcal{B}_s)}{s^2}\right]_1^r - 2\int_1^r \frac{\text{Area}(\mathcal{B}_s(p))}{s^3}\, ds\right) \\
&= \frac{1}{(\ln r)^2}\left(\left(\frac{\text{Area}(\mathcal{B}_r(p))}{r^2} - \text{Area}(\mathcal{B}_r(p))\right) - 2\cdots\right)
\end{aligned}$$

As $r \to \infty$, the first term goes to 0. Since energy is nonnegative and only the first term is positive, we get that it goes to 0.

We've proven that if $\Sigma \subseteq \mathbb{R}^3$ is stable and complete without boundary, then $\text{Area}(\mathcal{B}_r(p)) \le \frac{4}{3}\pi r^2$. We'd like to prove $\sup |A|^2 \le cr^{-2}$. If Σ doesn't have any boundary, then this holds for all r. Taking $r \to \infty$,

$$|A|^2(p) \le \sup_{\mathcal{B}_{\frac{r}{2}}(p)} |A|^2 \le cr^{-2} \to 0.$$

(This result is by Schoen in 1982.) Then the second fundamental form is 0 at every point, so it must be a plane: The derivative of the normal is 0, so the normal is constant; hence the surface must be in a plane orthogonal to this constant normal. We get $|A|^2 \equiv 0$, so n is constant, and $\Sigma = n^\perp$.

Theorem 24.2 (Bernstein Theorem, 1911)**:** If $\Sigma \subseteq \mathbb{R}^3$ is a stable minimal surface in $\mathbb{R}^3$ without boundary, then Σ is a plane.

Proof. Let $\Sigma \subseteq \mathbb{R}^3$ be stable. Then $\text{Area}(\mathcal{B}_r(p)) \le \frac{4}{3}\pi r^2$. Then there exist ϕ_r so that $\int |\nabla\phi_r|^2 \to 0$, ϕ_r has compact support, and $\phi_r = 1$ on $B_1(p)$.

Now the inequality $0 \le -\int_\Sigma \phi L\phi$ ($L\phi = \Delta_\Sigma\phi + |A|^2\phi$) becomes, after integrating by parts and using the fact that ϕ is compactly supported,

$$\begin{aligned}
\int \phi L\phi &= \int \phi(\Delta\phi + |A|^2\phi) \\
&= -\int |\nabla\phi|^2 + \int |A|^2\phi^2
\end{aligned}$$

Since Σ is stable,

$$\int |\nabla\phi|^2 \ge \int |A|^2\phi^2.$$

Inserting ϕ_r, we get (noting ϕ_r is 1 on the unit ball)

$$\int |\nabla \phi_r|^2 \geq \int |A|^2 \phi_r^2 \geq \int_{\mathcal{B}_1(p)} |A|^2.$$

Now the LHS goes to 0, so $|A|^2(p) = 0$. This proves the Bernstein Theorem. $\square$

The catenoid is the surface of revolution of hyperbolic cosine. Topologically it is a cylinder. It is complete without boundary. The catenoid can't be stable, because if it were stable it would have to be a plane. The same goes for the helicoid.

How do we prove the more general statement? This is quite useful.

Theorem 24.3 (Schoen, 1982)**:** thm:schoen Let $\Sigma \subseteq \mathbb{R}^3$ be stable. Suppose $\mathcal{B}_r(p) \subseteq \Sigma \backslash \partial\Sigma$. Then

$$\sup_{\mathcal{B}_{\frac{r}{2}}(p)} |A|^2(p) \leq cr^{-2}$$

for some constant c independent of p and r.

We use the following.

Theorem 24.4 (Choi-Schoen)**:** thm:choi-schoen There exists an $\varepsilon > 0$ such that if $p \in \Sigma \subseteq \mathbb{R}^3$ is a minimal surface and $\mathcal{B}_r(p) \subseteq \Sigma \backslash \partial\Sigma$, then $\int_{\mathcal{B}_r(p)} |A|^2 < \varepsilon$ implies $\sup_{\mathcal{B}_{\frac{r}{2}}} |A|^2 \leq r^{-2}$.

This is key.

Proof of Theorem (24.3) *using Theorem* (24.4). Imagine we have a ball $\mathcal{B}_r(p)$. We can scale it so the ball is very large. Now $\int_{\mathcal{B}_r(p)} |A|^2$ is invariant under scaling. (The second fundamental form goes down and area goes up; they cancel each other out.)

We just need to prove the bound for the second fundamental form in the center. Then you can do it everywhere.

But if the ball is very large, we can find a function ϕ that is 1 on the unit ball and has very small area. We use quadratic area bounds proved in this setting. We have $\int |\nabla \phi|^2 < \varepsilon$ and by stability,

$$\int |\nabla \phi|^2 \geq \int |A|^2 \phi^2 \geq_{\mathcal{B}_1} |A|^2.$$

The integral is small because of point mass bounds (from Choi-Shoen), so we have the theorem. We used the quadratic area bound. $\square$

These theorems are examples of a general type of theorem common in nonlinear differential equations. The second fundamental form is like energy. If you have an energy inequality, then you get a point mass estimate. (If the energy is above a certain threshold, then we don't get an estimate.)

Consider the second fundamental form on the catenoid: we have $\int |A|^2 < \infty$. This condition is called "finite total curvature." This is not small, so we don't have a pointwise estimate. On the other hand , the helicoid is not bounded.

As $|A|^2 = -2K$, the condition $\int |A|^2 < \infty$ is called "finite total curvature."

You can find all this in [2].

Proof sketch of Theorem 24.4. We first show Simon's inequality

$$\Delta |A|^2 \geq -2|A|^4.$$

Let $u : \mathbb{R}^n \to \mathbb{R}$. Consider $I(r) = r^{1-n} \int_{\partial B_r(0)} u$. We have by Stokes's Theorem

$$I'(r) = r^{1-n} \int_{\partial B_r(0)} \frac{du}{ds} = r^{1-n} \int_{\mathcal{B}_r(0)} \Delta u.$$

If u is harmonic this is constant. Then $I(r) = \lim_{s\to 0} I(s) = \mathrm{Vol}(\partial B_1) u(0)$. We have a mean value equality

$$u(0) = \frac{1}{\mathrm{Vol}(\partial B_r(0))} \int_{\partial B_r(0)} u.$$

If u is subharmonic, we get inequality in one direction, if supharmonic, we get inequality in other direction. If we have an eigenfunction (or subsolution) $\Delta u + \lambda u = 0$ we get a mean value equality, with some constant depending on λ, r.

Simon's inequality is a nonlinear inequality. There's a simple way of arguing by contradiction so we can actually assume $|A|^2(p) = 1$. We get $\sup_{\mathcal{B}_r(p)} |A|^2 \leq 4$. Then trivially we can replace one $|A|^2$ by 4, and now have a linear inequality. The inequality says $|A|$ is a subsolution to an eigenvalue equation. Using the mean value inequality, A at the center is bounded by the mean. But we assumed $\int |A|^2$ is really small. Thus we get contradiction. The reduction is simple, and nothing to do with minimal surface; it's just a calculus fact about functions.

$$\Delta |A|^2 \geq -2|A|^4 \geq -2|A|^2|A|^2 \geq -8|A|^2.$$

Now locate the point where $F = (r-d)^2 |A|^2$ is maximal. □

Lecture 24
References

[1] A. Besse. *Manifolds all of whose geodesics are closed.*

[2] T. Colding and W. Minicozzi. *A Course in Minimal Surfaces*, volume 121 of *Graduate Studies in Mathematics.* AMS, 2011.

[3] M. do Carmo. *Riemannian Geometry.* Birkhäuser, 1992.