Measure Theory

Lectures delivered by D. Stroock
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Contents

Introduction

D. Stroock taught a course (18.125) on Measure Theory at MIT in Spring 2011. These are my "live-TEXed" notes from the course. The template is borrowed from Akhil Mathew.

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Lecture 1
Wed. 2/2/2011

§1 Riemann integration

To integrate a function $f : J \to R$, where $J = [a_1, b_1] \times \cdots \times [a_N, b_N]$, take a non-overlapping cover $\mathcal{C}$ of J by nonoverlapping rectangles (i.e. for distinct $I, I' \in \mathcal{C}$, $I^\circ \cap I'^\circ = \phi$). Let $\xi \in \Xi(\mathcal{C})$ be a choice function that assigns to each I an element in I ($\xi(I) \in I$). Let

$$\mathcal{R}(f; \mathcal{C}, \xi) = \sum_{I \in \mathcal{C}} f(\xi(I)) \mathrm{vol}(I)$$

where $\mathrm{vol}(I)$ is the product of its sides.

One says that f is Riemann integrable if there exists $A \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\mathcal{R}(f, \mathcal{C}, \xi) - A| < \epsilon \text{ for all } \mathcal{C} \text{ with } ||\mathcal{C}|| < \delta, \xi \in \Xi(\mathcal{C}).$$

where

$$||\mathcal{C}|| = \max_I \mathrm{diam}(I).$$

This value of A is denoted by

$$A = (R) \int_J f(x)\, dx$$

Theorem 1.1: Any continuous function is Riemann integrable.

Proof. Uniform continuity of f (from compactness of domain) gives that approximations get close; completeness of $\mathbb{R}$ gives existence of A. □

Lemma 1.2: Suppose that $\mathcal{C}$ is any collection of rectangles I.

1. If $\mathcal{C}$ is non-overlapping and $J \supseteq \bigcup \mathcal{C}$, then $\mathrm{vol}(J) \ge \sum_{I \in \mathcal{C}} \mathrm{vol}(I)$.
2. If $J \subseteq \bigcup \mathcal{C}$, then $\mathrm{vol}(J) \le \sum_{I \in \mathcal{C}} \mathrm{vol}(I)$.

Proof. Without loss of generality, we may assume $J \subseteq \bigcup \mathcal{C}$ (just intersect rectangles with J), and $I^\circ \ne \phi$ for any $I \in \mathcal{C}$.

Induct on number of dimensions N. Consider $N = 1$. Let $I = [a_I, b_I]$.

For the first part, choose $a_J \le c_0 < \cdots < c_l \le b_J$ such that

$$\{c_k : 0 \le k \le l\} = \{a_I : I \in \mathcal{C}\} \cup \{b_I : I \in \mathcal{C}\}.$$

Let $\mathcal{C}_k = \{I \in \mathcal{C} : [c_{k-1}, c_k] \subseteq I\}$. Note

1. $\mathrm{vol}(I) = \sum_{k, I \in \mathcal{C}_k} (c_k - c_{k-1}) = b_I - a_I$.

2. If $\mathcal{C}$ is non-overlapping, then I is in at most one $\mathcal{C}_k$ (by definition of c_i as endpoints).

Then

$$\sum_{I\in\mathcal{C}} \text{vol}(I) = \sum_{I\in\mathcal{C}} \sum_{k:I\in\mathcal{C}_k} (c_k - c_{k-1}) \le \sum_{k=1}^{l} \sum_{I\in\mathcal{C}_k} (c_k - c_{k-1}) \le c_l - c_0 \le b_J - a_J = \text{vol}(J).$$

For the second part, if $J = \bigcup \mathcal{C}$ then $c_0 = a_J$, $c_l = b_J$, and $\mathcal{C}_k \neq \phi$ for any $1 \le k \le l$. (For this second assertion, consider 2 cases: c_k is the left or right hand endpoint. Argument is the same. For the right endpoint, choose I so that $b_I \ge c_k$ and $a_I \le a_{I'}$ for every I' such that $b_{I'} \ge c_k$—i.e. left-hand endpoint is as small as possible. Then $a_I \le c_{k-1}$; else any interval starting at c_{k-1} ends before c_k, contradiction.) Now

$$\sum_{I\in\mathcal{C}} \text{vol}(I) = \sum_{I\in\mathcal{C}} \sum_{k:I\in\mathcal{C}_k} (c_k - c_{k-1}) \ge \sum_{k=1}^{l} (c_k - c_{k-1}) = b_J - a_J.$$

When $N > 1$, we can write $I = R_I \times [a_I, b_I]$ where R_I is a $(n-1)$-dimensional rectangle. Apply a similar argument, but with $R_J = \bigcup_{I\in\mathcal{C}_k} R_I$. □

To "remove" the choice function we consider the Riemann upper and lower sums.

$$\mathcal{U}(f;\mathcal{C}) = \sum_{I\in\mathcal{C}} (\sup_I f)\,\text{vol}(I) \ge \mathcal{R}(f;\mathcal{C},\xi)$$
$$\mathcal{L}(f;\mathcal{C}) = \sum_{I\in\mathcal{C}} (\inf_I f)\,\text{vol}(I) \le \mathcal{R}(f;\mathcal{C},\xi)$$

Proposition 1.3: Let $f : J \to \mathbb{R}$ be bounded. f is Riemann integrable if and only if

$$\lim_{||\mathcal{C}||\to 0} \mathcal{L}(f;\mathcal{C}) = \lim_{||\mathcal{C}||\to 0} \mathcal{U}(f;\mathcal{C}).$$

Proof. "⇐"—squeeze theorem. "⇒"—choose choice function so close to upper/lower sum. □

The lemma applies when $\mathcal{C}_2$ is a refinement of $\mathcal{C}_1$, written $\mathcal{C}_1 \le \mathcal{C}_2$ (every rectangle of $\mathcal{C}_2$ is inside a rectangle in $\mathcal{C}_1$). Then $\mathcal{U}(f;\mathcal{C}_1) \ge \mathcal{U}(f;\mathcal{C}_2)$ and $\mathcal{L}(f;\mathcal{C}_1) \le \mathcal{L}(f;\mathcal{C}_2)$. Since I_1 is covered by nonoverlapping intervals of $\mathcal{C}_2$; $\text{vol}(I_1)$ is sum of volumes of those intervals.

Lecture 2
Fri. 2/4/2011

§1 Riemann integrability

Theorem 2.1: Let $f : J \to \mathbb{R}$ be bounded. Then

1. $\lim_{||\mathcal{C}|| \to 0} \mathcal{U}(f, \mathcal{C}) = \inf_{\mathcal{C}} \mathcal{U}(f, \mathcal{C})$.
2. $\lim_{||\mathcal{C}|| \to 0} \mathcal{L}(f, \mathcal{C}) = \sup_{\mathcal{C}} \mathcal{L}(f, \mathcal{C})$.
3. f is Riemann integrable if and only if

$$\inf_{\mathcal{C}} \mathcal{U}(f, \mathcal{C}) = \sup_{\mathcal{C}} \mathcal{L}(f, \mathcal{C}).$$

   where the infimum and supremum are taken over all finite exact nonoverlapping coverings.

Proof. We use the following.

Lemma 2.2: Given $\mathcal{C}$ and $\varepsilon > 0$ there exists δ such that $||\mathcal{C}'|| \le \delta$ such that $\mathcal{U}(f, \mathcal{C}') \le U(f, \mathcal{C}) + \varepsilon$. (Note $\mathcal{C}'$ need not be a refinement.)

Similarly, there exists δ such that $||\mathcal{C}'|| \le \delta$ such that $\mathcal{L}(f, \mathcal{C}') \ge L(f, \mathcal{C}) - \varepsilon$. (Note $\mathcal{C}'$ need not be a refinement.)

Proof. Consider $I' \in \mathcal{C}'$. Then either

1. $I' \subseteq I$ for $I \in \mathcal{C}$ (the "good" type) or
2. I' hits an edge (the "bad" case).

The terms in the first case do not cause a problem—if every I' were of this type then $\mathcal{U}(f, \mathcal{C}') \le \mathcal{U}(f, \mathcal{C}')$.

The rectangles in the second case cannot have a large combined area for $||\mathcal{C}'||$ small—they must be in a δ-neighborhood of the edges. In fact

$$\left| \sum_{I'} (\sup_{I'} f) \operatorname{vol}(I') \right| \le 2\delta ||f||_u C$$

where C depends on N, the cardinality of $\mathcal{C}$, and J, and the uniform norm is

$$||f||_u = \sup_{x \in J} |f(x)|.$$

□

Choose $\mathcal{C}$ so the upper sum is close to the infimum:

$$\mathcal{U}(f, \mathcal{C}) \le \inf_{\mathcal{C}} \mathcal{U}(f, \mathcal{C}) + \frac{\varepsilon}{2}.$$

Then find δ as in the lemma (for $\frac{\varepsilon}{2}$); for $||\mathcal{C}|| < \delta$, we have

$$\mathcal{U}(f, \mathcal{C}) \le \inf_{\mathcal{C}} \mathcal{U}(f, \mathcal{C}) + \varepsilon.$$

Item 2 follows similarly.

Use Proposition 1.3 to get item 3. □

§2 Riemann-Stieltjes integral

In the Riemann integral we integrate with respect to "homogeneous density", dx means summing $b_I - a_I$. For the Riemann integral we replace dx with $d\psi$, and sum $\psi(b_I) - \psi(a_I)$ instead of $b_I - a_I$.

Definition 2.3: The Riemann sum of ϕ over $\mathcal{C}$ with respect to ψ relative to ξ is

$$\mathcal{R}(\varphi|_\psi, \mathcal{C}, \xi) = \sum_{I \in \mathcal{C}} \varphi(\xi(I)) \Delta_I \psi, \quad \Delta_I \psi = \psi(b_I) - \psi(a_I).$$

ϕ is Riemann integrable with respect to ψ if there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $||\mathcal{C}|| \le \delta$ and any ξ,

$$|\mathcal{R}(\varphi|\psi, \mathcal{C}, \xi) - A| < \varepsilon.$$

Then we write

$$(R) \int_J \varphi(x) d\psi(x) = A.$$

Proposition 2.4: For $\varphi \in \mathcal{C}(J, \mathbb{R}), \psi \in C^1(J, \mathbb{R})$,

$$(R) \int_J \phi(x)\, d\psi(x) = (R) \int_J \varphi(x) \psi'(x)\, dx.$$

Proof. By the Mean Value Theorem,

$$\psi(b_I) - \psi(a_I) = \psi'(\eta(I)) \operatorname{vol}(I).$$

Now use uniform continuity of ψ'. □

Example 2.5: Suppose $a = a_0 < a_1 < \ldots < a_n = b$, and ψ is constant on (a_{m-1}, a_m) for $m = 1, \ldots, n$ (a "step" function with a few naughty points), and $\varphi \in C(J; \mathbb{R})$. Then

$$(R) \int_J \varphi\, d\psi = \sum_{m=1}^{n-1} \varphi(a_m)(\psi(a_m+) - \psi(a_m-)) + \varphi(a)(\psi(a+) - \psi(a)) + \varphi(b)(\psi(b) - \psi(b-)).$$

Proof. Consider an interval $I \in \mathcal{C}$. We may assume $\mathcal{C}$ has a fine enough mesh so no interval contains more than one a_i. Then either

1. $I \cap \{a_0, \dots, a_n\}$, i.e. $I \subseteq (a_{m-1}, a_m)$ for some m. ("Good case") Then $\Delta_I \psi = 0$.
2. $a_m \in I^\circ$. Then $\Delta_I \psi = \psi(a_m+) - \psi(a_m-)$.
3. $a_m = a_I$ or b_I. Then

$$\Delta_I \psi = \begin{cases} \psi(a_m+) - \psi(a_m), & a_m = a_I \\ \psi(a_m) - \psi(a_m-), & a_m = b_I \end{cases}.$$

□

Example 2.6:

$$(R) \int_J (\alpha\varphi_1 + \beta\varphi_2)\, d\psi = \alpha \int_J \phi_1 \, d\psi + \beta \int_J \varphi_2 \, d\psi.$$

Example 2.7: Let $J = J_1 \cup J_2$ and $J_1^\circ \cap J_2^\circ = \varphi$. Then

$$(R) \int_J \varphi \, d\psi = (R) \int_{J_1} \varphi \, d\psi + (R) \int_{J_2} \varphi \, d\psi.$$

Proof. We want

$$|\mathcal{R}(\varphi|\psi, \mathcal{C}_1, \xi_1) - \mathcal{R}(\varphi|\psi, \mathcal{C}_1', \xi_1')| < \varepsilon$$

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\xi = \xi_1 \cup \xi_2$, and similarly with $\mathcal{C}'$ and ξ'. The difference above equals

$$|\mathcal{R}(\varphi|\psi, \mathcal{C}, \xi) - \mathcal{R}(\varphi|\psi, \mathcal{C}', \xi')|$$

as needed.

Choose $\mathcal{C}_1, \mathcal{C}_2$ so the Riemann sums of the RHS integrals are close to the integral; then glue as above. □

Lecture 3
Mon. 2/7/2011

§1 Integration by parts

Theorem 3.1 (Integration by parts)**:** Suppose φ and ψ are bounded functions on J. If φ is Riemann integrable, then ψ is φ-Riemann integrable, and

$$(R) \int_J \psi(x) \, d\xi(x) = [\varphi(x)\psi(x)]_a^b - (R) \int_J \varphi(x) \, d\psi(x).$$

Proof. Let $\mathcal{C} = \{[\alpha_{m-1}, \alpha_m] : 1 \leq leqn\}$ where $a = \alpha_0 < \ldots < \alpha_n = b$. Take $\varphi([\alpha_{m-1}, \alpha_m]) = \beta_m \in [\alpha_{m-1}, \alpha_m]$. Now

$$\begin{aligned}\mathcal{R}(\varphi|\psi; \mathcal{C}, \varphi) &= \sum_{m=1}^{n} \psi(\beta_m)(\varphi(\alpha_m) - \varphi(\alpha_{m-1})) \\ &= \sum_{m=1}^{n} \psi(\beta_m)\varphi(\alpha_m) - \sum_{m=0}^{n-1} \psi(\beta_{m+1})\varphi(\alpha_m) \\ &= \psi(\beta_n)\varphi(b) - \psi(\beta_1)\varphi(a) - \sum_{m=1}^{n-1} \varphi(\alpha_m)(\psi(\beta_{m+1}) - \psi(\beta_m))\end{aligned}$$

Now we think of the β_m as the endpoints of intervals and α_m as the choices. Let $\beta_0 = a$ and $\beta_{n+1} = b$. Rearranging gives

$$\psi(b)\varphi(b) - \psi(a)\varphi(a) - \sum_{m=0}^{n} \varphi(\alpha_m)(\psi(\beta_{m+1}) - \psi(\beta_m)).$$

The mesh size of the β-partition is at most twice the mesh size of the α-partition. Since φ is ψ-integrable, this last sum approaches $(R)\int_J \varphi(x)\, d\psi(x)$, as needed. □

Corollary 3.2 (Fundamental Theorem of Calculus)**:** Suppose φ is differentiable. Then

$$\int_J \frac{d\xi(x)}{dx}\, dx = [\varphi(x)]_a^b$$

Proof. Take $\psi = 1$ and note $\int_J \frac{d\xi(x)}{dx}\, dx = \int_J d\xi(x)$. □

§2 Riemann-Stieltjes integrability

Theorem 3.3: If ψ is increasing, then every $\varphi \in C(J; \mathbb{R})$ is ψ-Riemann integrable.

Proof. Define $\mathcal{U}(\varphi|\psi; \mathcal{C})$ and $\mathcal{L}(\varphi|\psi; \mathcal{C})$. Use uniform continuity. □

Proposition 3.4: If ψ_1, ψ_2 are increasing, then for every $\varphi \in C(J; R)$,

$$(R)\int_J \varphi(x)\, d(\psi_1 - \psi_2) = (R)\int_J \varphi(x)\, d\psi_1 - (R)\int_J \varphi(x)\, d\psi_2.$$

§3 Variation

Proposition 3.5: If $\psi = \psi_2 - \psi_1$,

$$\left|(R)\int_J \varphi(x)\,d\psi(x)\right| \leq ||\varphi||_u(\Delta_J\psi_1 + \Delta_J\psi_2).$$

We can ask the following: for what functions ψ does there exist K_ψ such that for every $\varphi \in C(J;R)$ ψ-integrable and

$$\left|(R)\int \varphi\,d\psi\right| \leq K_\psi ||\varphi||_u?$$

Theorem 3.6: Only functions that are the difference of two increasing functions.

We give a better description of this criterion.

Definition 3.7: Let

$$S(\psi, \mathcal{C}) = \sum_{I\in\mathcal{C}} |\Delta_I\psi|.$$

The **variation** of ψ on J is

$$\text{Var}(\psi, J) = \sup_{\mathcal{C}} S(\psi;\mathcal{C}).$$

This measures the amount of "up-and-down" jiggliness of the function.

Proposition 3.8 (Basic properties)**:**

1. By the Triangle Inequality, if $\mathcal{C}'$ is a refinement of $\mathcal{C}$, then $S(\psi;\mathcal{C}') \geq S(\psi;\mathcal{C})$.
2. If $J = J_1 \cup J_2$, $\text{Var}(\psi, J) = \text{Var}(\psi, J_1) + \text{Var}(\psi, J_2)$.

Definition 3.9: For $a \in \mathbb{R}$ let $a_+ = \max(a, 0)$ and $a_- = \max(-a, 0)$. Define

$$S_+(\psi;\mathcal{C}) = \sum_{I\in\mathcal{C}} (\Delta_I\psi)^+$$
$$S_-(\psi;\mathcal{C}) = \sum_{I\in\mathcal{C}} (\Delta_I\psi)^-$$
$$\text{Var}_\pm(\psi;\mathcal{C}) = \sup_{\mathcal{C}} S_\pm(\psi;\mathcal{C}).$$

Note $a_+ - a_- = a$ and $a_+ + a_- = |a|$, so

$$S_+(\psi,\mathcal{C}) - S_-(\psi,\mathcal{C}) = \Delta_J\psi$$
$$S_+(\psi,\mathcal{C}) + S_-(\psi,\mathcal{C}) = S(\psi;\mathcal{C})$$
$$S_+(\psi,\mathcal{C}) = \frac{1}{2}(\Delta_J\psi - S(\psi;\mathcal{C}))$$
$$S_-(\psi,\mathcal{C}) = \frac{1}{2}(\Delta_J\psi + S(\psi;\mathcal{C}))$$

If approach extreme values for one than for all of them. Statements pass to variations.

$$\begin{aligned}\mathrm{Var}_+(\psi; J) - \mathrm{Var}_-(\psi; J) &= \Delta_J \psi \\ \mathrm{Var}_+(\psi, J) + \mathrm{Var}_-(\psi; J) &= \mathrm{Var}(\psi; J)\end{aligned}$$

Lecture 4
Wed. 2/9/2011

§1 Functions of Bounded Variation

Theorem 4.1: A function can be written as the difference of two increasing functions if and only if it has bounded variation.

Proof. Suppose that

$$\mathrm{Var}(\psi; [a, b]) < \infty.$$

Then

$$\begin{aligned}\psi(x) - \psi(a) &= V_+(\psi; [a, x]) - V_-(\psi; [a, x]) \\ \psi(x) &= [\psi(a) + V_+(\psi; [a, x])] - V_-(\psi; [a, x])\end{aligned}$$

□

§2 Convergence rate

Let $f : [0, 1] \to \mathbb{R}$ be smooth ($f \in C^1$) Let

$$R_n(f) = \frac{1}{n} \sum_{m=1}^{n} f\left(\frac{m}{n}\right).$$

$$(R) \int_0^1 f(x)\, dx = \lim_{n\to\infty} R_n(f).$$

How fast does $R_n(f)$ converge to the integral? In general we can't do better than the following argument:

$$\begin{aligned}(R) \int_0^1 f(x)\, dx - R_n(f) &= \sum_{m=1}^{n} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \left(f(x) - f\left(\frac{m}{n}\right)\right) \\ &= \int_{\frac{m-1}{n}} \left(x - \frac{m}{n}\right) f'(\xi_x)\, dx \\ &\le n \cdot ||f'||_u \frac{1}{n^2} = \frac{1}{n} ||f'||_u\end{aligned}$$

This is the best we can do because letting $f(x) = x$,

$$R_n(f) = \frac{1}{n}\sum_{m=1}^{n}\frac{m}{n} = \frac{n(n+1)}{2n^2} = \frac{1}{2} + \frac{1}{2n}.$$

However, if f is periodic we can do a lot better. We integrate by parts, in such a way so that we don't boundary terms, by choosing constants of integration appropriately. Then

$$\begin{aligned}
&(R)\int_0^1 f(x)\,dx - R_n(f)\\
&= -\sum_{m=1}^{n}(R)\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left(f(x) - f\left(\frac{m-1}{n}\right)\right)dx\\
&= -\sum_{m=1}^{n}(R)\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left(x - \frac{m-1}{n}\right)f'(x)\,dx\\
&= -\sum_{m=1}^{n}(R)\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left(x - \frac{m-1}{n} - \frac{1}{2n}\right)f'(x)\,dx \qquad \text{by periodicity}\\
&= -\sum_{m=1}^{n}(R)\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left(x - \frac{m-1}{n} - \frac{1}{2n}\right)\left(f'(x) - f\left(\frac{m}{n}\right)\right)dx.
\end{aligned}$$

We used periodicity to subtract off the average value of $x - \frac{m-1}{n}$ on $\left[\frac{m-1}{n}, \frac{m}{n}\right]$. (Note $\int_0^1 cf'(x)\,dx = 0$.) In the last step we used $\frac{1}{2n}$ is the average value of $x - \frac{m-1}{n}$ on the integral, so $\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left(x - \frac{m-1}{n} - \frac{1}{2n}\right) = 0$. The last expression is bounded by $\frac{1}{2n^2}||f''||_u$, better than $\frac{1}{n}$.

Now assume that f and f' are both periodic C^1. (i.e. $f'(1) = f'(0)$ as well)

$$\begin{aligned}
&\sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left(x - \frac{m-1}{n} - \frac{1}{2n}\right)\left(f'(x) - f\left(\frac{m}{n}\right)\right)dx\\
&= -\frac{1}{2n}\sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}} f'(x) - f'\left(\frac{m}{n}\right) + \sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left(x - \frac{m-1}{n}\right)\left(f'(x) - f'\left(\frac{m}{n}\right)\right)\\
&= -\frac{1}{2n}\sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}} f'(x) - f'\left(\frac{m}{n}\right) + \sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left(x - \frac{m-1}{n}\right)^2 f''(x)
\end{aligned}$$

Use periodicity to subtract off average value.

$$-\frac{1}{2n}\sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}} f'(x) - f'\left(\frac{m}{n}\right) + \sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}}\left[\left(x - \frac{m-1}{n}\right)^2 - \frac{1}{3n^2}\right]\left(f''(x) - f''\left(\frac{m}{n}\right)\right)dx.$$

Repeating this process. If f and all its derivatives are periodic (of period 1) then the error in the Riemann approximation is going to 0 faster than any $\frac{1}{n^k}$.

To define Riemann integals for complex valued-functions, just look at real and complex parts separately. Let $f(x) = e^{2\pi i x}$. Let $\xi_{m,n} = \frac{m}{n}\left(1 - \frac{1}{n}\right)$. Then

$$n\left(\int_0^1 e^{2\pi i x}\,dx - \frac{1}{n}\sum_{m=1}^n f(\xi_{m,n})\right) = \frac{1}{n}\sum_{m=1}^n e^{\frac{2\pi i(1-\frac{1}{n})m}{n}} = e^{\frac{2\pi i(1-\frac{1}{n})}{n}}\frac{1-e^{-\frac{2\pi i}{n}}}{1-e^{\frac{2\pi i(1-\frac{1}{n})}{n}}} \to -1.$$

Moral: Periodicity destroyed by bad choice function.

Definition 4.2: The **Bernoulli numbers** b_l are inductively defined by

$$b_{l+1} = \sum_{k=0}^{l} \frac{(-1)^k}{(k+2)!} b_{l-k}.$$

§3 Lebesgue integration: Motivation

Let E be a set. Let $f : E \to \mathbb{R}$. Let μ be "volume" or measure of Γ; that is μ is defined on nice subsets of E. We want to find a way to integrate f with respect to μ. We want to find a partition of E into subsets Γ so that f is constant or close to constant on each set Γ. We then add up $\mu(E)y$ (y this constant value).

Riemann: E has topological structure and f is nice with respect to topology (e.g. continuous). Partition into sets small from the topological standpoint, then give me f, it'll be nearly constant on each subset.

But if f doesn't respect topology this FAILS!

Lebesgue: Look at sets of the form

$$\{x | f(x) \in [(m-1)2^{-n}, m2^{-n}]\}$$

f is nearly constant on each of these subsets, regardless of topological niceness. Now integrate.

But first we need to find the "volume" or "measure" of these sets! They will be HIDEOUS... Integration theory is easy compared to assigning measures.

Lecture 5
Fri. 2/11/2011

§1 Measure

For a set $E \neq \phi$ define the power set

$$\mathcal{P}(E) = 2^E = \{\Gamma : \Gamma \subseteq E\}.$$

Definition 5.1: A subset $\mathcal{B} \subseteq \mathcal{P}(E)$ is a **σ-algebra** it satisfies the following properties:

1. $E \in \mathcal{B}$.
2. $\mathcal{B}$ is closed under complementation: $\Gamma \in \mathcal{B}$ implies $\Gamma^c = E \backslash \Gamma \in \mathcal{B}$.
3. $\{\Gamma_n : n \geq 1\} \subseteq \mathcal{B}$ implies $\bigcup_{n=1}^{\infty} \Gamma_n \in \mathcal{B}$.

(If item 2 is satisfied just for finite instead of countable unions then we call $\mathcal{B}$ an algebra.)

Note that items 2 and 3 imply that a countable intersection of elements in $\mathcal{B}$ is in $\mathcal{B}$, and a difference of sets in $\mathcal{B}$ is in $\mathcal{B}$.

Definition 5.2: We call $(E, \mathcal{B})$ is a measurable space. A measure on $(E, \mathcal{B})$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that

1. $\mu(\phi) = 0$.
2. (Countable additivity) If $\{\Gamma_n : n \geq 1\}$ is a family of pairwise disjoint subsets of E, then
$$\mu\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) = \sum_{n=1}^{\infty} \mu(\Gamma_n),$$
i.e. the volume of the whole is the sum of the volume of the parts.

Compare this to the definition of a topological space—measurable spaces have measureable sets while topologies have open sets.

Example 5.3: Define a measure μ on the integers $\mathbb{Z}$ by associating some $\mu_i \geq 0$ for each integer i, and setting
$$\mu(\Gamma) = \sum_{i \in \Gamma} \mu_i.$$

Our strategy is to start with some class of nice, well-defined subsets, and generate more.

Definition 5.4: For a family of subsets $\mathcal{C} \subseteq \mathcal{P}(E)$, define the σ-algebra generated by $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$, to be the smallest σ-algebra containing $\mathcal{C}$. In other words it is the intersection of all σ-algebras containing $\mathcal{C}$. (This is well-defined since the power set is a σ-algebra containing $\mathcal{C}$.)

If E is a topological space and $\mathcal{C} = \{\Gamma \subseteq E : \Gamma \text{ open}\}$ then $\sigma(\mathcal{C}) = \mathcal{B}_E$ is called the **Borel σ-algebra**.

Lebesgue showed that there exists a unique Borel **σ-algebra** on $\mathcal{B}_{\mathbb{R}_N}$ such that $\mu_{\mathbb{R}^N}(I) = \text{vol}(I)$.

§2 Basic results

Proposition 5.5: 1. If $A \subseteq B$ are sets in $\mathcal{B}$ then $\mu(A) \leq \mu(B)$.

2. (Countable subadditivity) Let $\{\Gamma_n : n \geq 1\} \subseteq \mathcal{B}$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) \leq \sum_{n=1}^{\infty} \mu(\Gamma_n).$$

(The sets are not necessarily disjoint, so the RHS counts "overlap.")

3. A countable union of subsets of measure zero has measure 0.

4. We write $B_n \nearrow B$ if $B_1 \subseteq B_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} B_n = B$.
If $B_n \nearrow B$ then $\mu(B_n) \nearrow \mu(B)$ (i.e. $\mu(B_n) \to \mu(B)$ from below).

5. We write $B_n \searrow B$ if $B_1 \supseteq B_2 \supseteq \cdots$ and $\bigcap_{n=1}^{\infty} B_n = B$.
If $B_n \searrow B$ *and* $\mu(B_1) < \infty$ then $\mu(B_n) \searrow \mu(B)$ (i.e. $\mu(B_n) \to \mu(B)$ from above).

Proof. 1. Note $B \backslash A \in \mathcal{B}$. Hence

$$\mu(B) = \mu(A) + \mu(B \backslash A) \geq \mu(A).$$

2. Let

$$B_n = \Gamma_n \backslash \bigcup_{m=1}^{n-1} \Gamma_m.$$

Then by countable additivity,

$$\mu\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(\Gamma_n).$$

In the last step we used $B_n \subseteq \Gamma_n$ and part 1.

3. Follows directly from part 2.

4. Like in part 2, take $A_n = B_n \backslash B_{n-1}$. Then

$$\mu(B_n) = \sum_{m=1}^{n} \mu(A_m) \nearrow \sum_{m=1}^{\infty} \mu(A_m) = \mu(B).$$

5. By the previous part, $B_1 \backslash B_n \nearrow B_1 \backslash B$ giving $\mu(B_1 \backslash B_n) \nearrow \mu(B_1 \backslash B)$.
Now use $\mu(B \backslash A) = \mu(B) - \mu(A)$, which holds because $\mu(B) < \infty$.

□

Note item 5 is false without the assumption that $\mu(B_1) < \infty$. For example, consider the measure on $\mathbb{Z}$ with $\mu(\Gamma) = |\Gamma|$, and take $B_n = \{i : i \geq n\}$.

Note from item 3, the existence of Lebesgue measure implies $\mathbb{R}$, or any interval of $\mathbb{R}$, is uncountable, since all countable subsets have measure 0 and any interval does not.

Definition 5.6: We say $\mathcal{C}$ is a Π-system if $\mathcal{C}$ is closed under intersection, i.e. if $A \in \mathcal{C}$ and $B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$.

Lecture 6
Mon. 2/14/2011

§1 More on σ-algebras

We give another characterization of $\sigma(\mathcal{C})$, the smallest σ-algebra containing $\mathcal{C}$.

Definition 6.1: We say that $\mathcal{H}$ is a Λ-system if

1. $E \in \mathcal{H}$.
2. If $A, B \in \mathcal{H}$ and $A \cap B = \phi$ then $A \cup B \in \mathcal{H}$.
3. If $A, B \in \mathcal{H}$ and $A \subseteq B$ then $B \backslash A \in \mathcal{H}$.
4. If $\{A_n : n \geq 1\} \subseteq \mathcal{H}$ and $A_n \nearrow A$ then $A \in \mathcal{H}$.

Theorem 6.2: Suppose $\mathcal{C}$ is a Π-system with $\mathcal{C} \subseteq \mathcal{P}(E)$. Let μ, ν on $\sigma(\mathcal{C})$ be such that $\mu(E) = \nu(E) < \infty$ and $\mu(A) = \nu(A)$ for all $A \in \mathcal{C}$. Then $\mu(A) = \nu(A)$ for all $A \in \sigma(C)$.

If two measures agree on the whole set E and a Π-system, then they agree on the smallest σ-algebra generated by the Π-system. (cf. Two continuous functions equal on a dense set are equal on the whole set.)

Proof. The set of subsets $\mathcal{H}'$ on which μ and ν agree satisfy conditions 1 (by assumption) and 2 (by additivity). It satisfies condition 3 because $\mu(B\backslash A) = \mu(B) - \mu(A)$ (measures are finite). It satisfies condition 4 by Proposition 5.5(4). Hence $\mathcal{H}'$ is a Λ-system. (This is the motivation for the definition of a Λ-system.)

It suffices to show that

$$\sigma(\mathcal{C}) = \bigcap \{\mathcal{H} : \mathcal{H} \text{ is } \Lambda\text{-system containing } \mathcal{C}\} =: \mathcal{H}_0.$$

(In other words $\sigma(C)$ is the smallest Γ-system containing $\mathcal{C}$.)

We first show that $\mathcal{H}_0$ is a σ-algebra.

Lemma 6.3: $\mathcal{B}$ is a σ-algebra iff $\mathcal{B}$ is both a Π and Λ-system.

Proof. The forward direction is clear. For the reverse direction, take $B = E$ in condition 3 to see $\mathcal{B}$ is closed under complementation. If $A, B \in \mathcal{B}$ then $A \cup B \in \mathcal{B}$, since we can write $A \cup B$ as a union of disjoint sets in $\mathcal{B}$ and use condition 2 as follows:

$$A \cup B = A \cup (B \backslash (A \cap B)).$$

Thus (by induction) $\mathcal{B}$ is closed under finite union. Now consider $\{A_n : n \geq 1\} \subseteq \mathcal{B}$. Then $\bigcup_{m=1}^{\infty} A_m \nearrow \bigcup_{m=1}^{\infty} A_m$ so by condition 4, $\bigcup_{m=1}^{\infty} A_m \subseteq \mathcal{B}$, and $\mathcal{B}$ is closed under countable union. □

Now $\mathcal{H}_0$ is a Λ-system because it is the intersection of a family of Λ-systems. Now

$$\mathcal{H}_1 = \{\Gamma \subseteq E : \Gamma \cap A \subseteq \mathcal{H}_0 \text{ for every } A \in \mathcal{C}\}$$

is a Λ-system (check it!). Since $\mathcal{C}$ is a Π-system, $\mathcal{C} \subseteq \mathcal{H}_1$ and hence $\mathcal{H}_1 \supseteq \mathcal{H}_0$ ($\mathcal{H}_0$ being the smallest Λ-system containing $\mathcal{C}$). This gives

$$\Gamma \cap A \in \mathcal{H}_0 \text{ for every } \Gamma \in \mathcal{H}_0, \Delta \in \mathcal{C}. \tag{1}$$

Let

$$\mathcal{H}_2 = \{\Gamma \subseteq E : \Gamma \cap A \in \mathcal{H}_0 \text{ for every } A \in \mathcal{H}_0\}.$$

Then $\mathcal{H}_2$ is a Λ-system; it contains $\mathcal{C}$ by (1). Hence $\mathcal{H}_0 \subseteq \mathcal{H}_2$, and $\mathcal{H}$ is a Π-system. □

Given $(E, \mathcal{B}, \mu)$, can we extend the measure to an even larger σ-algebra? Yes.

Definition 6.4: Define the completion of B with respect to μ as

$$\bar{\mathcal{B}}^{\mu} = \{\Gamma \subseteq E : \text{there exist } A, B \in \mathcal{B}, A \subseteq \Gamma \subseteq B, \mu(B \backslash A) = 0\}.$$

We can define a measure $\bar{\mu}$ on $\bar{\mathcal{B}}^{\mu}$ defined by

$$\bar{\mu}(\Gamma) = \mu(A).$$

(This is well-defined because if $A_i \subseteq \Gamma \subseteq B_i$ and $\mu(B_i \backslash A_i) = 0$ for $i = 1, 2$ then $\mu(A_1) \leq \mu(B_2) = \mu(A_2) \leq \mu(B_1) = \mu(A_1)$ so $\mu(A_1) = \mu(A_2)$.) Then $(E, \bar{\mathcal{B}}^{\mu}, \bar{\mu})$ is called the **completion** of $(E, \mathcal{B}, \mu)$.

This is again a σ-algebra: Indeed $A \subseteq \Gamma \subseteq B$ implies $B^c \subseteq \Gamma^c \subseteq A^c$ with $\mu(A^c \backslash B^c) = \mu(B \backslash A)$. Similarly it's closed under countable union.

Definition 6.5: Let $\mathcal{G}(E)$ denote the open sets of the topological space E, and let $\mathcal{B} = \sigma(\mathcal{G}(E))$ be the Borel algebra with measure μ. $\Gamma \subseteq E$ is μ**-regular** if for every $\varepsilon > 0$ there exists $F \in \mathcal{F}(E)$ such that $G \in \mathcal{G}(E)$, $F \subseteq \Gamma \subseteq G$ and $\mu(G \backslash F) < \varepsilon$.

Restrict choice of bread: Upper slice is open and bottom slice is closed. But we're more lenient about the middle: it doesn't have to be 0, just less than ε.

Proposition 6.6: A regular set is in the completion.

Proof. Take $G_n \supseteq \Gamma \supseteq F$ with the property that $\mu(G_n \backslash F_n) \le \frac{1}{n}$. Without loss generality we may assume that the G_n are decreasing. (Replace G_n with $\bigcap_{m=1}^{n} G_m$.) Similarly we may assume that F_n are increasing. Let

$$D = \bigcap_{n=1}^{\infty} G_n, \quad C = \bigcup_{n=1}^{\infty} F_n$$

D is not necessarily open and C is not necessarily closed but both are Borel set. Hence they are in $\mathcal{B}$ (as countable intersections/complements of elements in $\mathcal{B}$ are in $\mathcal{B}$, and open sets are in $\mathcal{B}$. □

Given a topology E, let $G_\delta(E)$ be the set of countable intersections of open sets, and let $F_\sigma(E)$ be the set of countable unions of closed sets. If E is a metric space, the open sets are in $F_\sigma(E)$. closed under ctable unions Clsoed sets are in $G_\delta(E)$. countable intersections $F_{\sigma\delta}(E)$=take countable unions of elements in $F_\sigma(E)$. Ad infinitum. Beyond countably infinitely many times, get all Borel sets.

Lecture 7
Wed. 2/16/2011

§1 Constructing measures

Let (E, ρ) be a metric space, and $\mathcal{R} \subseteq \mathcal{P}(E)$ be a family of compact subsets. Let V be a map $V : \mathcal{R} \to [0, \infty)$. (Keep in mind the model that $E = \mathbb{R}^N$, $\mathcal{R}$ is the set of closed rectangles, and V the volume of the rectangles.) Suppose the following hold.

1. $\mathcal{R}$ is a Π-system, i.e. if $I, J \in \mathcal{R}$ then $I \cap J \in \mathcal{R}$.

2. $\phi \in \mathcal{R}$ and $V(\phi) = 0$.

3. If $I \subseteq J$ then $V(I) \le V(J)$.

4. Suppose $\{I_1, \ldots, I_n\} \in \mathcal{R}$ and $J \in \mathcal{R}$.

   (a) If $J \subseteq \bigcup_{m=1}^{n} I_m$ then $V(J) \le \sum_{m=1}^{n} V(I_m)$.

   (b) If $J \supseteq \bigcup_{m=1}^{n} I_m$ and the I_m's are non-overlapping then $V(J) \ge \sum_{m=1}^{n} V(I_m)$.

5. For all $I \in \mathcal{R}$ and all $\varepsilon > 0$, there exist $I, I' \in \mathcal{R}$ such that $I'' \subseteq I^\circ$, $I \in I'^\circ$ and $V(I') \le V(I'') + \varepsilon$.

6. For all open $G \in \mathcal{G}(E)$ there exists a sequence $\{I_n : n \ge 1\} \subseteq \mathcal{R}$ nonoverlapping sets such that $G = \bigcup_{m=1}^{\infty} I_n$.

Proposition 7.1: These properties hold for $E = \mathbb{R}^N$, $\mathcal{R}$ is the set of closed rectangles, and V the volume of the rectangles.

Proof. Item 4 holds by Lemma 1.2. Item 5 holds since we can enlarge or shrink $\mathcal{R}$ by a tiny bit. For item 6, consider a checkerboard of cubes of side length 2^{-n}:

$$\{k2^{-n} + [0, 2^{-n}]^N : k \in \mathbb{Z}^N\}$$

Take $m < n$ and $Q \in \mathcal{C}_m$, $Q' \in \mathcal{C}_n$. Either

1. The interiors of Q, Q' intersect, and $Q' \subseteq Q$, or
2. The interiors of Q, Q' do not intersect.

I.e. "either Q' is a descendant of Q or they are unrelated."

We use a greedy algorithm to stuff cubes in G. We show that in fact item 6 can be done with cubes of arbitrarily small length.

By splitting G into bounded parts we may assume G is bounded. Let $\delta > 0$ be given.

Let $G_0 = G$ and define the $G_k, \mathcal{A}_k$ inductively as follows. Let n_k be the smallest n such that there exist $Q \in \mathcal{C}_n$ for $Q \subseteq G_k$ and $2^{-n} < \delta$.

Let $\mathcal{A}_k = \{Q \in \mathcal{C}_{n_k} : Q \subseteq G\}$ and

$$G_k = G_{k-1} \backslash \bigcup_{Q \in \mathcal{C}_{n_k}} Q.$$

Let

$$\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n.$$

It is clear that $\mathcal{A} \subseteq G$. Take a point $x \in G$. Take n sufficiently large; there's a cube from $\mathcal{C}_n$ such that $x \in \mathcal{C}_n$. Either that cube was chosen or one of its ancestors (which contains that cube) was chosen. □

Our goal is to prove the following.

Theorem 7.2: Given conditions (1)–(6), there exists a Borel measure such that $\mu(I) = V(I)$ for all $I \in \mathcal{R}$.

Proof. We will proceed in the following steps.

1. Define $\tilde{\mu}$ for all sets $\Gamma \subseteq E$ by

$$\tilde{\mu}(\Gamma) = \inf \left\{ \sum_{n=1}^{\infty} V(I_n) : I_n \in \mathcal{R} \text{ and } \Gamma \subseteq \bigcup_{m=1}^{\infty} I_m \right\}.$$

   Then $\tilde{\mu}$ is subadditive (Lemma 7.3).

2. $\tilde{\mu}$ of a countable family of nonoverlapping sets J_l of $\mathcal{R}$ is just the sum of the volumes $V(J_l)$. (A generalization of the fact that $\tilde{\mu}$ agrees with V.) (Lemma 7.4)

3. Give an alternate characterization of $\tilde{\mu}$ (Proposition 8.1):

$$\tilde{\mu}(\Gamma) = \inf\{\tilde{\mu}(G) : G \in \mathcal{G}(E) \text{ and } \Gamma \subseteq G\}.$$

4. Let $\mathcal{L}$ be the collection of $\Gamma \subseteq E$ such that for every $\varepsilon > 0$ there exists $G \supseteq \Gamma$ with $\tilde{\mu}(G \backslash \Gamma) < \varepsilon$. Then $\mathcal{L}$ is a σ-algebra (Theorem 8.2).

5. $\mu = \tilde{\mu}|\mathcal{L}$ is a measure on $\mathcal{L}$ (Thereom 8.4).

□

Lemma 7.3: $\tilde{\mu}$ is sub-additive, i.e.

$$\tilde{\mu}\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) \leq \sum_{n=1}^{\infty} \tilde{\mu}(\Gamma_n).$$

Proof. This is another application of the $2^{-m}\varepsilon$ trick. Given $\varepsilon > 0$, for each m, choose $\{I_{m,n} : n \geq 1\} \subseteq \mathcal{R}$ such that

$$\Gamma_m \subseteq \bigcup_{n=1}^{\infty} I_{m,n} \text{ and } \sum_{n=1}^{\infty} V(I_{m,n}) \leq \tilde{\mu}(\Gamma_m) + 2^{-m}\varepsilon.$$

The collection $\{I_{m,n} : (m,n) \in \mathbb{N}^2\}$ covers $\bigcup_{m=1}^{\infty} \Gamma_m$.

Since all terms are nonnegative, we can write the sum as an iterated sum.

$$\sum_{(m,n)\in\mathbb{N}^2} V(I_{m,n}) = \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} V(I_{m,n}) \leq \sum_{m=1}^{\infty} \left(\tilde{\mu}(\Gamma_m) + \varepsilon 2^{-m}\right) \leq \sum_{m=1}^{\infty} \tilde{\mu}(\Gamma_m) + \varepsilon.$$

The lemma follows upon combining this with

$$\sum_{(m,n)\in\mathbb{Z}} V(I_{m,n}) \geq \tilde{\mu}\left(\bigcup_{m=1}^{\infty} I_{m,n}\right).$$

□

Now we look for a σ-algebra $\mathcal{B}_\mu \subseteq \mathcal{B}_E$ such that $\tilde{\mu}$ on $\mathcal{B}_\mu$ is a measure. We need to check that $\tilde{\mu}(I) = V(I)$. In fact we check the following stronger statement.

Lemma 7.4: Let $\{J_1, \ldots\} \subseteq \mathcal{R}$ be a set of nonoverlapping rectangles. Then

$$\tilde{\mu}\left(\bigcup_{l=1}^{\infty} J_l\right) = \sum_{l=1}^{\infty} V(J_l).$$

Proof. First we check that equality holds when there are a finite number of J_l, $1 \le l \le L$. Note "$\le$" holds because $\tilde{\mu}$ is the infimum over all covers and J_l form a cover for $\bigcup J_l$. We need to show "$\ge$." Let $\{I_m : m \ge 1\}$ with $\bigcup_{l=1}^L J_l \subseteq \bigcup_{m=1}^\infty I_m$. Choose I'_m containing I_m in its interior so that $V(I'_m) \le V(I_m) + \varepsilon 2^{-m}$ (same trick!). Now $\bigcup_{l=1}^L J_l$ is compact as it is a finite union of compact sets. Since

$$\bigcup_{m=1}^{\infty} I_m^\circ \supseteq \bigcup_{l=1}^{L} J_l,$$

we can take a finite subcover of $\bigcup J_l$:

$$\bigcup_{m=1}^{n} I_m^\circ \supseteq \bigcup_{l=1}^{L} J_l.$$

Consider the cover $I_{l,m} = J_l \cap I'_m$ (so we can change order of summation). Note $I_{1,m}, \ldots, I_{L,m}$ are nonoverlapping and $I'_m \supseteq \bigcup_{l=1}^L I_{l,m}$. Then by condition 4,

$$\left(\sum_{m=1}^{n} V(I_m)\right) + \varepsilon \ge \sum_{m=1}^{n} V_m(I'_m) \overset{4}{\ge} \sum_{m=1}^{n}\sum_{l=1}^{L} V(I_{l,m}) \overset{4}{\ge} \sum_{l=1}^{L} V(J_l).$$

Now take the infimum to get $\tilde{\mu}\left(\bigcup J_l\right)$ on the left-hand side.

Now we extend to the infinite case. We want

$$\tilde{\mu}\left(\bigcup_{l=1}^{\infty} J_l\right) = \sum_{l=1}^{\infty} V(J_l)$$

Note "$\le$" holds because $\tilde{\mu}$ is the infimum over all covers and J_l form a cover. We need to show "$\ge$." Use the finite case

$$\tilde{\mu}\left(\bigcup_{l=1}^{n} J_l\right) \ge \sum_{l=1}^{n} V(J_l)$$

and take $n \to \infty$ to get the equation above. □

Lecture 8
Fri. 2/18/2011

§1 Constructing measures, continued

We continue to assume all 6 conditions given in the previous lecture.

By assumption 6, given $G \in \mathcal{G}(E)$ we can write $G = \bigcup_{m=1}^{\infty} I_m$. Lemma 7.4 says

$$\tilde{\mu}(G) = \sum_{m=1}^{\infty} V(I_m).$$

It immediately follows that if $G \cap G' = \phi$ then

$$\tilde{\mu}(G \cup G') = \tilde{\mu}(G) + \tilde{\mu}(G').$$

(Just put the two families of rectangles together.)

Proposition 8.1:

$$\tilde{\mu}(\Gamma) = \inf\{\tilde{\mu}(G) : G \in \mathcal{G}(E) \text{ and } \Gamma \subseteq G\}.$$

Proof. Now "$\leq$" obviously holds.

We need to show that given $\bigcup_{m=1}^{\infty} I_m \supseteq \Gamma$ and $\varepsilon > 0$,

$$\tilde{\mu}(G) \leq \sum_{m=1}^{n} V(I_m) + \varepsilon.$$

Take I'_m so that $I'^{\circ}_m \supseteq I_m$ and $V(I'_m) \leq V(I_m) + 2^{-m}\varepsilon$. Now take

$$G = \bigcup_{m=1}^{\infty} I'^{\circ}_m.$$

□

We want a family of sets in $\mathcal{P}(E)$ that is a σ-algebra, such that the restriction of $\tilde{\mu}$ there is countably additive and hence a measure.

Let $\mathcal{L}$ be the collection of $\Gamma \subseteq E$ such that for every $\varepsilon > 0$ there exists $G \supseteq \Gamma$ with $\tilde{\mu}(G \backslash \Gamma) < \varepsilon$.

Theorem 8.2: $\mathcal{L}$ is a σ-algebra.

Proof.

1. Every open set is in $\mathcal{L}$:

$$\mathcal{G}(E) \subseteq \mathcal{L}.$$

2. $\mathcal{L}$ is closed under countable unions, by the $2^{-n}\varepsilon$ argument.

3. "Sets of measure 0" are in $\mathcal{L}$:

$$\tilde{\mu}(\Gamma) = 0 \implies \Gamma \in \mathcal{L}.$$

(Take an open set $U \supseteq \Gamma$ so that $\tilde{\mu}(\Gamma) \leq \varepsilon$.)

4. Compact sets are in $\mathcal{L}$. We use the following lemma:

Lemma 8.3: Suppose $K, K' \subset\subset E$ with $K \cap K'$ and $K \cap K' = \phi$. (The notation $\subset\subset$ means "compact subset of".) Then

$$\tilde{\mu}(K \cup K') = \tilde{\mu}(K) + \tilde{\mu}(K').$$

Proof. "$\leq$" holds by subadditivity.

We can find disjoint open subsets G, G' containing K, K'. Let H be an open set such that $H \supseteq K \cup K'$. Then

$$\begin{aligned}\tilde{\mu}(H) &\geq \tilde{\mu}((G \cap H) \cup (G' \cap H)) \\ &\overset{7.4}{=} \tilde{\mu}(G \cap H) + \tilde{\mu}(G' \cap H) \\ &= \tilde{\mu}(K) + \tilde{\mu}(K')\end{aligned}$$

Taking the infimum of the LHS gives $\tilde{\mu}(K \cup K')$ by Lemma 8.1. □

Now we show that if $K \subset\subset E$ then $K \in \mathcal{L}$. We need to show for $\varepsilon > 0$ there exists $G \supseteq K$ so $\tilde{\mu}(G \backslash K) < \varepsilon$. We claim $\tilde{\mu}(K) < \infty$. By assumption 6, we can write $K = \bigcup_{m=1}^{\infty} I_m$. For each I_m we can choose open I'_m so that $I'^{\circ}_m \supseteq I_m$ with $V(I'_m) \leq V(I_m) + 1$. We can choose a finite cover: $K \subseteq \bigcup_{m=1}^{n} I'^{\circ}_m$. Then $\tilde{\mu}(K) \leq \sum_{m=1}^{n} V(I'^{\circ}_m)$.

We can choose $G \supseteq K$ so that $\tilde{\mu}(G) \leq \tilde{\mu}(K) + \varepsilon$, i.e. $\tilde{\mu}(G) - \tilde{\mu}(K) \leq \varepsilon$. Look at $G \backslash K$; it is open. (K is compact in a metric space, so closed.) Thus by assumption 6, we can write $G \backslash K = \bigcup_{m=1}^{n} I_m$. Now we show

$$\tilde{\mu}\left(K \cup \bigcup_{m=1}^{n} I_m\right) \leq \tilde{\mu}(G)$$

Now $K \cup \bigcup_{m=1}^{n} I_m$ is compact because it is a finite union of compact sets. They are disjoint so by the Lemma 7.4 the volume is

$$\tilde{\mu}(K) + \sum_{m=1}^{n} V(I_m).$$

Hence

$$\tilde{\mu}(G \backslash K) = \sum_{m=1}^{\infty} V(I_m) \leq \varepsilon.$$

So K is in $\mathcal{L}$.

5. Closed sets F are in $\mathcal{L}$. Indeed, write $E = \bigcup_{m=1}^{\infty} I_m$ (assumption 6). Given closed F, $F_n = G \cap \bigcup_{m=1}^{n} I_m$ is compact (because a closed set in a compact set is compact) and hence in $\mathcal{L}$ by item 3. Now $F = \bigcup_{n=1}^{\infty} F_n$, so by item 2 ($\mathcal{L}$ closed under countable union), $F \in \mathcal{L}$.

6. Countable unions of closed sets are in $\mathcal{L}$, i.e. $\mathcal{F}_\sigma(E) \subseteq \mathcal{L}$: Use item 5 and item 2 ($\mathcal{L}$ closed under countable union).

7. If $\Gamma \in \mathcal{L}$ then $\Gamma^c \in \mathcal{L}$. Given $\Gamma \in \mathcal{L}$, choose $G_n \supseteq \Gamma$ so that $\tilde{\mu}(G_n \backslash \Gamma) \leq \frac{1}{n}$. Let $D = \bigcap_{n=1}^\infty G_n$. Then $D \in \mathcal{G}_\delta(E)$, $D \supseteq \Gamma$, and $\tilde{\mu}(D \backslash \Gamma) = 0$. Hence $D \backslash \Gamma \in \mathcal{L}$ by item 3.

   Now $\Gamma^c \backslash D^c = D \backslash \Gamma$. So

   $$\Gamma^c = D^c \cup (\Gamma^c \backslash D^c) \in \mathcal{L}.$$

   Note $D^c \in \mathcal{F}_\sigma(E)$ so $D^c \in \mathcal{L}$ by item 6. Hence $D \in \mathcal{F}_\sigma \subseteq \mathcal{L}$.

We've shown the three defining properties of a σ-algebra (Definition 5.1) in items 1 (E is open), item 2 (closed under countable union), and item 7 (closed under complements). □

Theorem 8.4: $\mu = \tilde{\mu}|\mathcal{L}$ is a measure on $\mathcal{L}$.

Proof. We need to show μ is countably additive. Since $\Gamma^c \in \mathcal{L}$, given $\Gamma \in \mathcal{L}$ and $\varepsilon > 0$ there exists $F \in \mathcal{F}(E), F \subseteq \Gamma$ so that $\tilde{\mu}(\Gamma \backslash F) < \varepsilon$ by the same trick.

Assume we have $\Gamma_n, n \geq 1$ mutually disjoint, relatively compact (i.e. having compact closure) sets in $\mathcal{L}$. We first prove countable additivity in this case. Given $K_n \subset\subset E$, for $K_n \subseteq \Gamma_n$. Thus

$$\mu\left(\bigcup_{m=1}^\infty \Gamma_m\right) \geq \mu\left(\bigcup_{m=1}^n \Gamma_m\right) \geq \mu\left(\bigcup_{m=1}^n K_m\right) = \sum_{m=1}^n \mu(K_m).$$

Now take K_n such that $\mu(\Gamma_n) \leq \mu(K_n) + \varepsilon 2^{-n}$ and take $n \to \infty$.

The opposite inequality holds by subadditivity of $\tilde{\mu}$.

For the general case, choose $I_m \in \mathcal{R}$ so that $E = \bigcup_{n=1}^\infty I_n$, and let $A_1 = I_1$, $A_{n+1} = I_{n+1} \backslash \bigcup_{m=1}^n I_m$. Then the A_n's are mutually disjoint, relatively compact sets in $\mathcal{L}$, and so are $\Gamma_{m,n} = A_m \cap \Gamma_n$. Now use the previous part on the $\Gamma_{m,n}$. □

Theorem 8.5 (Uniqueness)**:**