Real Analysis

Contents

1 Ordered Fields

1-1 Ordered sets and fields

Let S be an ordered set and let $E \subseteq S$. E is bounded below, above if there exists $a, b \in S$ (called a lower or upper bound) such that $a < x, x < b$ for all $x \in E$, respectively.

If a is a lower bound such that any $c < a$ is not a lower bound for E, then a is the **greatest lower bound (supremum)** of E, denoted by $\sup X$. The supremum is unique when it exists. Similarly, if b is an upper bound such that any $c > b$ is not an upper bound for E, then b is the **least upper bound (infimum)** of E, denoted by $\inf E$.

S has the **least upper bound property** if whenever $E \subseteq S$ is nonempty and bounded above, $\sup E$ exists in S. This is equivalent to the greatest lower bound property.

An **ordered field** is a field that is an ordered set satisfying:

1. If $y < z$ then $x + y < x + z$.
2. If $x > 0, y > 0$ then $xy > 0$.

An ordered field F is **Archimedean** if for all $x, y \in F$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$. $\mathbb{Q}$ and $\mathbb{R}$ are both Archimedean.

1-2 Construction of the Reals 1: Dedekind Cuts

There exists a unique ordered field $\mathbb{R}$ (the **real numbers**) with the least upper bound property; it contains $\mathbb{Q}$ as a subfield.

Pf.

1. The real numbers are associated with subsets $a \in \mathbb{Q}$ (called cuts) satisfying:
   a. $a \neq \phi, \mathbb{Q}$.
   b. If $p \in a, q \in \mathbb{Q}$ and $q < p$, then $q \in a$. (If a contains p, it contains all numbers less than p.)
   c. If $p \in a$ then $p < r$ for some $r \in a$. (No matter which $p \in a$ we choose, we can always find $r > p$ in a larger than it.)
2. We say $a < b$ if $a \subset b$.
3. $\mathbb{R}$ has the LUB property.
4. Let $a + b = \{r + s | r \in a, s \in b\}$. Verify the axioms for addition. The inverse of a is $b = \{p | \exists r > 0, -p - r \notin a\}$ (some rational number smaller than $-p$ is not in a).
5. Show that $b < c \Rightarrow a + b < a + c$.
6. For positive a, b, let $ab = \{p | p \leq rs \text{ for some } r \in a, s \in b; r, s > 0\}$.
7. Complete the definition by defining multiplication involving negative elements. Verify the remaining axioms.
8. Each rational number q is associated with $\{x \in \mathbb{Q} | x < q\}$. Check that with this embedding, the rational numbers are an ordered subfield.

In the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, every nonempty subset has a infimum and supremum.[1]

[1] The definitions of limit, etc. extend to numbers in $\overline{\mathbb{R}}$ is we let the neighborhoods of ∞ be all sets of the form $\{x | x > L\}$, and similarly for $-\infty$.

2 Metric Spaces

2-1 Metric Spaces

A set X with a real-valued function (a metric) $d(p,q)$ on pairs of points in X is a **metric space** if:

1. $d(p,q) \geq 0$ with equality iff $p = q$.
2. $d(p,q) = d(q,p)$
3. $d(p,q) \leq d(p,r) + d(r,q)$ (Triangle inequality)

Ex.

- Discrete space: For any set X, define the metric

$$d(x,y) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$$

- **N-dimensional Euclidean space** $\mathbb{R}^n$, with distance defined as

$$d\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

- $\mathbb{R}_p^n, p \geq 1$:

$$d(x,y) = \left(\sum_{i=1}^{n} |x_k - y_k|^p\right)^{\frac{1}{p}}$$

  - To prove this is a metric, use Hölder's Inequality…

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}, p \geq 1$$

  - …to derive Minkowski's Inequality:

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}}, \frac{1}{p} + \frac{1}{q} = 1; p, q > 1$$

$\mathbb{R}_0^n$

$$d(x,y) = \max_{1 \leq k \leq n} |x_k - y_k|$$

- $C_{[a,b]}^2$: Continuous functions defined on $[a,b]$

$$d(f,g) = \left(\int_a^b [x(t) - y(t)]^2\right)^{\frac{1}{2}}$$

- $C_{[a,b]}$:

$$d(f,g) = \max_{a \leq t \leq b} |f(t) - g(t)|$$

- l_p: Infinite sequences with $\sum_{i=1}^{\infty} x_i^p < \infty$:

$$d(x,y) = \left(\sum_{i=1}^{\infty} (x_k - y_k)^p\right)^{\frac{1}{p}}$$

- m: Bounded infinite sequences

$$d(x,y) = \sup_k |x_k - y_k|$$

2-2 Definitions

Term	Definition in metric space X	Definition in topology X
neighborhood	For $\varepsilon > 0$, the ε-**neighborhood** of a point p is the set $N_\varepsilon(p) = \{p \in X \mid d(p,q) < \varepsilon\}$	A **neighborhood** of p is an open set containing p.
contact point	p is a **contact point** of $E \subseteq X$ if every neighborhood of p contains a point of E.	
limit point	p is a **limit point** of $E \subseteq X$ if every neighborhood of p contains a point of E besides p. E' is the set of limit points of E.	
isolated point	If $p \in E$ but is not a limit point of E, then p is an **isolated point**. (Contact points = limit points ∪ isolated points.)	
closed	E is **closed** if every limit point of E is in E.	E is **closed** if X-E is open.
closure	The **closure** of E is the set of contact points of E. $\bar{E} = [E] = E \cup E'$.	The **closure** $\bar{E} = [E]$ of X is the intersection of all closed sets contained in E.
interior point	p is an **interior point** if there is a neighborhood N of p such that $N \subseteq E$. The interior of N, denoted by $E^\circ = \operatorname{Int} E$, is the set of interior points (or union of open sets contained in E).	
open	E is **open** if every point of E is an interior point of E.	A **topology** on a set X is a collection $\mathcal{T}$ of subsets, called **open sets** satisfying: 1. $\phi, X \in \mathcal{T}$ 2. The union of an arbitrary collection of sets in $\mathcal{T}$ is in $\mathcal{T}$. 3. The intersection of a finite number of sets in $\mathcal{T}$ is in $\mathcal{T}$.
perfect	E is **perfect** if E is closed and every point of E is a limit point of E.	
bounded	E is **bounded** if there exists $M \in \mathbb{R}$ and $q \in X$ so that $d(p,q) < M$ for all $p \in E$.	N/A
dense	E is **dense** in X if every point of X is a contact point of E, i.e. $X = \bar{E}$.	

Closed sets satisfy the following:

1. X, ϕ are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

A metric space is a topology- the definitions in topology hold in a metric space. (Note that neighborhoods in metric spaces are more strictly defined.)

If p is a limit point of E, then every neighborhood of p contains infinitely many points of E. Thus a finite point set has no limit points.

%On closure:

1. $\bar{E}$ is closed.

	2. $E = \bar{E}$ iff E is closed. 3. $\bar{E} \subseteq F$ for every closed set F with $E \subseteq F$. Let E be a nonempty subset of $\mathbb{R}$ that is bounded above. If E is closed, then $\sup E \in E$. Subspace topology: Let $E \subseteq Y \subseteq X$. E is open relative to Y iff $E = Y \cap U$ for some open set U in X.
2-3	Separability A metric space is **separable** if it contains a countable[2] dense subset. *Ex.* $\mathbb{R}^k$ is separable since $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. m is not separable. Any subset of a separable space is separable. A **base** for a topology on X is a collection $\mathcal{B}$ of subsets, called base elements, of X such that 1. For each $x \in X$, there is at least one base element containing x. 2. If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $x \in B_3 \subseteq B_1 \cap B_2$ for some $B_3 \in \mathcal{B}$. In the topology generated by $\mathcal{B}$, a subset U is open if for each $x \in U$, there is a base element $B \in \mathcal{B}$ so that $x \in B \subseteq U$. In particular, each base element is open. Two other equivalent formulations: • $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$. • $\mathcal{B}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $C \in \mathcal{B}$ such that $x \in C \subset U$. Ex. ε-neighborhoods A metric space is separable iff it has a countable base. • Separable⇒Countable base: Let P be a countably dense subset and take $\{N_{1/n}(p) \| p \in P\}$. • Countable base⇒Separable: (true for any topology) Choose a point in each base element.
2-4	Compact Sets An **open cover** of a set E in a topology X is a collection $\mathcal{F}$ of open subsets such that $E \subseteq \bigcup_{G \in \mathcal{F}} G$. A subset $K \subseteq X$ is **compact** if every open cover of K contains a finite subcover. K is **sequentially (or countably) compact** if every infinite subset of K has a limit point in K. On subsets: • Suppose $K \subseteq Y \subseteq X$. Then K is compact relative to X if it is compact relative to Y. In other words, compactness is an intrinsic property. • Compact subsets are closed. • Closed subsets of compact sets are compact. Theorems on compact sets: • K is compact iff K is sequentially compact. ○ ⇒: Let E be infinite subset. If no limit points, each $m \in K$ has a neighborhood

[2] Here countable means finite or having same cardinality as $\mathbb{N}$.

containing at most 1 point of E- neighborhoods form a cover with no finite subcover.

- ⇐:
  - Sequentially compact⇒Separable: Given δ, choose any x_1, take x_{i+1} to be δ away from all $x_1, \dots, x_i$. This must stop. Let δ range over $1/n$; putting together x_i's gives countable dense subset.
  - Separable⇒Countable base
  - Countable base⇒Every cover has at most countable subcover: For $\mathcal{G}$ a base and $\mathcal{H}$ a subcover, associate each element $G \in \mathcal{G}$ contained in some $H \in \mathcal{H}$ with $f(G) \supseteq G$. $\cup f(G)$ is a finite subcover.
  - Sequentially compact⇒Nested nonempty sets $\{F_n\}$ have nonempty intersection: Take $x_n \in F_n$. If $E = \{x_n\}$ finite then done; else it has limit point, which is in the intersection.
  - Take countable subcover; let $F_n = \{x \in K, x \notin \bigcup_{i=1}^{n} G_i\}$. If $K \nsubseteq \bigcup_{i=1}^{n} G_i$ for any n, there is $y \in \bigcap_{i=1}^{\infty} F_n$ by above. Then $y \notin \bigcup_{i=1}^{\infty} G_i$, contradiction.

- If $\mathcal{F}$ is a family of compact subsets of X such that the intersection of every finite subcollection is nonempty, then $\bigcap_{K \in \mathcal{F}} K \neq \phi$.
- (Nested Intervals Theorem) If $\{I_n\}$ is a nested sequence of intervals ($I_n \supseteq I_{n+1}$), then $\bigcap_{n=1}^{\infty} I_n \neq \phi$. If the length of the intervals goes to 0, then the intersection consists of a single point.
- If $\{I_n\}$ is a nested sequence of k-cells (closed boxes in $\mathbb{R}^k$), then $\bigcap_{n=1}^{\infty} I_n \neq \phi$.
- Every k-cell is compact.
  - *Pf.* Suppose there's an open cover $\mathcal{F}$ without a finite subcover. Find a nested sequence $\{I_n\}$ of k-cells whose dimensions go to 0, such that the cells can't be covered by a finite subcollection of $\mathcal{F}$. Some point x is in $\bigcap_{n=1}^{\infty} I_n$. It's in an open set in $\mathcal{F}$ which is contained in I_n for n large enough, contradiction.

<u>Heine-Borel Theorem:</u> For a subset $E \subseteq \mathbb{R}^n$, the following properties are equivalent:

1. E is compact (every cover has a finite subcover).
2. E is closed and bounded.

<u>*Cor.*</u> Every bounded infinite subset of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.

2-5 Perfect Sets

Any nonempty perfect set in $\mathbb{R}^n$ is uncountable. Thus every interval $[a,b], a < b$ is uncountable.

The Cantor set: Let $E_0 = [0,1]$. Once E_i is defined, write it as a disjoint union of intervals in the form $[a,b]$, and replace each with $[a + \frac{1}{3}(b-a)] \cup [a + \frac{2}{3}(b-a), b]$ to form E_{i+1}. The **Cantor set** is $C = \bigcap_{n=1}^{\infty} E_n$. C is a (uncountable) perfect, compact set containing no segment. The Cantor set consists of all numbers whose ternary expansion consists only of the digits 0 and 2 (an infinite string of 2s being allowed).

2-6 Connected Sets

Two subsets A, B of a metric space X are **separated** if $\bar{A} \cap B = A \cap \bar{B} = \phi$. A subset E is **disconnected** if it is a union of two nonempty separated sets, and **connected** otherwise. Equivalent condition (see below): E is disconnected if there exist disjoint nonempty open

	A, B so that $X = A \cup B$. The union of sets in $\mathcal{F}$ is connected if every distinct pair of sets in $\mathcal{F}$ are not separated. For $x \in E$, the union of all connected subsets containing x is the **connected component** of X containing x. The connected components form a partition of E, and they are all closed sets. If X is a metric space with finitely many components, then the components are both closed and open (clopen). Conversely, any clopen set is a union of components of X. In particular, if X is connected, the only clopen sets are X and ϕ. In a **totally disconnected set**, all connected components are point sets. *Ex.* $\mathbb{Q}$ and the Cantor set C

3	Sequences and Series
3-1	Sequences and Convergence

3 Sequences and Series

3-1 Sequences and Convergence

Let $\{p_n\}_{n=1}^{\infty}$ be a **sequence** of points in a metric space X. The sequence **converges** to a point $p \in X$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n > N$, $d(p_n, p) < \varepsilon$. Else it **diverges**.

- $\{p_n\}_{n=1}^{\infty}$ converges to p if every open set containing p contains p_n for all but finitely many n. (This is the definition of convergence in a topological space.)
- If $\{p_n\}_{n=1}^{\infty}$ converges then it converges to a unique $p \in X$, denoted by $\lim_{n\to\infty} p_n$.
- If $\{p_n\}_{n=1}^{\infty}$ converges then it is bounded.
- If $E \subseteq X$, p is a contact point of E iff there exists a sequence $\{p_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} p_n = p$.

A **Cauchy sequence** is a sequence $\{p_n\}$ such that for every $\varepsilon > 0$, there is an integer N so that $d(p_n, p_m) < \varepsilon$ for all $m, n \geq N$. In other words, letting $E_N = \{p_N, p_{N+1}, \dots\}$ and defining $\operatorname{diam} S = \sup\{d(p,q) | p, q \in S\}$, $\lim_{N\to\infty} \operatorname{diam} E_N = 0$.
Cauchy Criterion: Every convergent sequence is Cauchy.

A sequence is **monotonically increasing, decreasing** if $a_n \leq a_{n+1}$, $a_n \geq a_{n+1}$, respectively. A monotonically increasing, decreasing sequence is convergent iff it is bounded above, below, respectively.

Basic properties (for $\mathbb{C}$): Suppose $\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t$.

- $\lim_{n\to\infty} s_n \pm t_n = s \pm t$
- $\lim_{n\to\infty} cs_n + d = cs + d$
- $\lim_{n\to\infty} s_n t_n = st$
- $\lim_{n\to\infty} 1/s_n = 1/s$, $s_n \neq 0$
- Squeeze Theorem: If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$.

The same properties and definitions hold if s_n, t_n are replace with functions defined on reals, letting the variable range over the reals.

A **subsequence** of $\{p_n\}$ is in the form $\{p_{n_i}\}$, where $n_1 < n_2 < \cdots$ are positive integers. The limits of subsequences are called subsequential limits.

- If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X. In particular, every bounded subsequence of $\mathbb{R}$^k contains a convergent subsequence.
- The subsequential limits for a closed subset.

Césaro-Stolz Lemma: Let $\{a_n\}, \{b_n\}$ be two sequences of real numbers and suppose either of the following holds.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ and $\{b_n\}$ is decreasing for sufficiently large n.
2. $\lim_{n\to\infty} b_n = \infty$ and $\{b_n\}$ is increasing for sufficiently large n.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$, provided the latter limit exists.

3-2 Lim inf and Lim sup

	Given a sequence $\{p_n\}$, let $E_N = \{p_N, p_{N+1}, \ldots\}$. Let $i_N = \inf E_N$ and $s_N = \sup E_N$. Define
	• $\lim_{n\to\infty} \inf p_n = \lim_{n\to\infty} i_n$
	• $\lim_{n\to\infty} \sup p_n = \lim_{n\to\infty} s_n$
	Properties:
	• Any sequence $\{p_n\}$ in $\mathbb{R}$ has a monotonic subsequence converging to $\lim_{n\to\infty} \inf p_n$, $\lim_{n\to\infty} \sup p_n$ (allowing $\infty, -\infty$).
	• Let S be the set of subsequential limit points (including $\pm\infty$). Then
	  ○ $\lim_{n\to\infty} \inf p_n = \inf S$
	  ○ $\lim_{n\to\infty} \sup p_n = \sup S$

3-3	Construction of the Reals 2: Cauchy Sequences 1. Identify the real numbers with equivalence classes of Cauchy sequences of rational numbers. Two sequences $\{a_n\}, \{b_n\}$ are equivalent if $\lim_{n\to\infty} a_n - b_n = 0$. Each rational number is associated with its constant sequence. 2. Define addition and multiplication as termwise addition and multiplication, and show it is well-defined. 3. For the multiplicative inverse, take the reciprocal of all terms, except those that are 0. (Sequence is eventually nonzero.) 4. Structure of ordered field: A real number is positive (greater than 0) if the sequence is eventually positive. $s > t$ if $s - t$ is positive. Check the order axioms. 5. $\mathbb{R}$ is Archimedean: We can find ε so the terms of a given positive $\{a_n\}$ are eventually at least ε. 6. $\mathbb{Q}$ is dense in $\mathbb{R}$: follows from construction. 7. $\mathbb{R}$ has the LUB property: Construct real sequences $\{u_n\}, \{l_n\}$ of upper and lower bounds of $\sup S$ so that $\lim_{n\to\infty} u_n - l_n = 0$ (you can make it halve each time). u_n, l_n approach the same real number, the sup. 8. $\mathbb{R}$ is complete: $\delta - \varepsilon$ funness.
3-4	Completion In a **complete** metric space, every convergent sequence is Cauchy. • Every compact metric space is complete. • Any Euclidean space $\mathbb{R}^n$ is complete. Each metric space X has a completion X^*: 1. The elements of X^* are equivalence classes of Cauchy sequences in X. Two sequences are equivalent if $\lim_{n\to\infty} d(p_n, q_n) = 0$. Each $p \in X$ is associated with a constant sequence. 2. Define distance by $\lim_{n\to\infty} d(p_n, q_n)$. 3. X^* is complete: $\delta - \varepsilon$ funness. 4. X is dense in X^* and $X = X^*$ if X is complete.
3-5	Infinite Series The partial sums of $\{a_n\}$ are $s_n = \sum_{k=1}^{n} a_k$. Define $$\sum_{k=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

	The sum converges if this limit exists; else it diverges. Infinite products are defined similarly.
	
	Convergence/ Divergence Tests:
	• Divergence Theorem: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.
	• A series of nonnegative terms converges iff its partial sums form a bounded sequence.
	• Basic Comparison Test: Suppose $a_n \geq b_n \geq 0$ for all $n \geq N$.
	    ○ (Convergence) If $\sum_{n=1}^{\infty} a_n$ converges then so does $\sum_{n=1}^{\infty} b_n$.
	    ○ (Divergence) If $\sum_{n=1}^{\infty} b_n$ diverges then so does $\sum_{n=1}^{\infty} a_n$.
	• Limit Comparison Test: Let $\{a_n\}, \{b_n\}$ be eventually positive sequences. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite and nonzero, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.
	• Ratio Test: Let $\{a_n\}$ be a sequence of nonzero terms.
	    ○ If $\limsup_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
	    ○ If $\liminf_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| \geq 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
	• Root Test: Let $l = \limsup_{n\to\infty} \sqrt[n]{\|a_n\|}$.
	    ○ If $l < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
	    ○ If $l > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges. If $\sqrt[n]{\|a_n\|} \geq 1$ for infinitely many distinct values of n then $\sum_{n=1}^{\infty} a_n$ diverges.
	• Cauchy's Condensation Criterion: Suppose $\{a_n\}$ is nonincreasing. $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.
	• P-Test: $\sum_{n=1}^{\infty} n^p$ converges iff $p < -1$.
	• Absolute Convergence: If $\sum_{n=1}^{\infty} \|a_n\|$ converges then $\sum_{n=1}^{\infty} a_n$ converges. It is said to **converge absolutely**.
	• Alternating Series (Leibniz) Test: If $a_n \geq a_{n+1}$ for all n and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. If the series converges but does not converge absolutely, it converges **conditionally**.
	    ○ Alternating Series Approximation Theorem: Suppose $\sum_{n=1}^{\infty} (-1)^n a_n$ satisfies the conditions above. Then the mth partial sum approximates the infinite series with an error of at most a_{m+1}:
	$$\left\|\sum_{n=m+1}^{\infty} (-1)^n a_n\right\| \leq a_{m+1}$$
	• Kummer's Test: Let $\{a_n\}, \{b_n\}$ be positive sequences. Suppose $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverges and let $x_n = b_n - \left(\frac{a_{n+1}}{a_n}\right) b_{n+1}$. Then $\sum_{n=1}^{\infty} a_n$ converges if $\liminf_{n\to\infty} x_n > 0$ and diverges if $x_n \leq 0$ for all n.
	For products:
	Coriolis Test: If $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} x_n^2$ converge, then so does $\prod_{n=1}^{\infty} (1 + x_n)$. (Pf. Take ln and use Taylor expansion.)
	
	Let $\sigma: \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$.
	• If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent to A then any rearrangement is also absolutely

	convergent to A. • If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent then for every $B \in \overline{\mathbb{R}}$, there exists a rearrangement that is conditionally convergent to B.[3] ○ Pf. Break up into a positive and negative sequence. Add terms from the positive sequence until sum overshoots B, add terms from the negative sequence until sum below B, and repeat.
3-6	Power Series A **power series** is in the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$. There exists $r \geq 0$ (possibly ∞), called the **radius of convergence**, so that 1. $f(x)$ converges for all complex $\|x\| < r$. 2. $f(x)$ diverges for all complex $\|x\| > r$. A **Laurent series** is in the form $f(x) = \sum_{n=k}^{\infty} a_n x^n, k > -\infty$.

[3] For complex number sequences, the set of possible sums is a point, a line, or the whole plane. (This is difficult.)

4	Limits and Continuity
4-1	Limits Let X and Y be metric spaces, $E \subseteq X$, and p be a limit point of E. Let $f: E \to Y$ be a function. The **limit** of f at p is $q \in Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in E$ with $0 < d(x,p) < \varepsilon$ we have $d(f(x), q) < \varepsilon$. $$\lim_{x\to p} f(x) = q$$ Note that $f(p)$ does not matter (it need not exist). Equivalently, for any sequence $\{x_n\} \subseteq E$ such that $x_n \neq p$ and $\lim_{n\to\infty} x_n = p$, we have $\lim_{n\to\infty} f(x_n) = q$. (This allows basic properties of sequences to carry over as below.) Infinite limits: (Definitions with $-\infty$ are similar.) • $\lim_{x\to\infty} f(x) = q$ if for every $\varepsilon > 0$ there exists L such that $\lvert f(x) - q\rvert < \varepsilon$ whenever $x > L$. • $\lim_{x\to a} f(x) = \infty$ if for every L there exists $\delta > 0$ such that $f(x) > L$ whenever $\lvert x - a\rvert < \delta$. The limit is unique if it is defined, and satisfies the following: Suppose $\lim_{x\to p} f(x) = L, \lim_{x\to p} g(x) = M$. • $\lim_{x\to p} f(x) \pm g(x) = L \pm M$ • $\lim_{x\to p} cf(x) + d = cL + d$ • $\lim_{x\to p} f(x)g(x) = LM$ • $\lim_{x\to p} f(x)/g(x) = L/M$, $M \neq 0$ • Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ for all x in a neighborhood of p except possibly at p, and $\lim_{x\to p} f(x) = L = \lim_{n\to\infty} h(x)$, then $\lim_{n\to\infty} g(x) = L$.
4-2	Continuity Let $E \subseteq X, f: E \to Y, p \in E$. f is **continuous** at p if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every x such that $d(x,p) < \varepsilon$, we have $d\big(f(x), f(p)\big) < \varepsilon$. f is continuous iff either p is an isolated point of E or $\lim_{x\to p} f(x) = f(p)$. Equivalently, for every sequence $\{x_n\}$ converging to p, $f(x_n)$ converges to $f(p)$. $f: X \to Y$ is continuous if f is continuous at every point $x \in X$. If $f: X \to Y$ and $g: Y \to Z$ are continuous then $g \circ f: X \to Z$ is continuous. (One topological definition) Let $f: X \to Y$ be any function between topological spaces. f is continuous iff for any open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open. Pf. 1. ⇒: For every open set U, $x \in f^{-1}(U)$ can find neighborhood of $f(x)$ in U. By continuity some neighborhood of x is in U. 2. ⇐: Take U equal to neighborhood of $f(x)$ of radius ε. $f^{-1}(U)$ is open and contains x; some neighborhood of x is in $f^{-1}(U)$. $f: X \to Y$ is continuous for every closed set $C \subseteq Y$, $f^{-1}(C)$ is closed in X. (Use $f^{-1}(E^c) = f^{-1}(E)^c$.

A **homeomorphism** is a continuous bijective function f such that $f^{-1}: Y \to X$ is continuous.

Basic properties:

- If f, g are continuous $f: X \to \mathbb{R}$, then $f+g, fg, \frac{f}{g}$ (if $g \neq 0$) are continuous.
- Let $f_1, \dots, f_n: X \to \mathbb{R}$. $f = (f_1, \dots, f_n)$ is continuous in $\mathbb{R}^n$ iff $f_1, \dots, f_n$ are continuous.
- $f(x) = d(x, p)$ is continuous.

Ex. Any polynomial $f: \mathbb{C}^n \to \mathbb{C}$ is continuous. Any rational function is continuous except at points where the denominator is 0.

4-3 Compactness and Uniform Continuity

Let $f: X \to Y$ be a continuous map of metric spaces, where X is compact. Then $f(X) \subseteq Y$ is also compact.
Pf. For an open cover of $f(X)$, take the inverse of each subset. They're open since f is continuous; choose a finite subcover and take the image.
Cor. Any continuous map $f: X \to \mathbb{R}^n$ from compact X is bounded.
Cor. (Weierstrass) A continuous function $f: X \to \mathbb{R}$ attains its maximum and minimum.

If X is compact and $f: X \to Y$ is continuous and bijective, then $f^{-1}: Y \to X$ is also continuous.
Pf. Take open $U \subseteq X$. $X - U$ is closed and hence compact (since X is compact). Then $f(X - U) = Y - f(U)$ (since f is bijective) is compact. Hence $f(U)$ is open.

A function $f: X \to Y$ is **uniformly continuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ (independent of x_1, x_2) such that $d\big(f(x_1), f(x_2)\big)$ for every $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$.

Heine-Cantor Theorem: A continuous function on a compact metric space is uniformly continuous.
Pf. Suppose else. For every $\varepsilon > 0$, we can find p_n, q_n so $d(p_n, q_n) < \frac{1}{n}$ but $d(f(p_n), f(q_n)) \geq \varepsilon$. By compactness, some subsequence p_{n_i} converges; then q_{n_i} converges to the same point p. They get arbitrarily close to p but $d(f(p_n), f(q_n)) \geq \varepsilon$, contradicting continuity at p.

If $f: X \to Y$ is continuous and X is connected then $f(X)$ is also connected.
Pf. If U_1, U_2 are open sets whose union is $f(X)$, then their inverses under f would be open sets whose union is X.

Intermediate Value Theorem:

1. Let $f: X \to \mathbb{R}$ be continuous on a connected metric space. If $f(x_1) < y < f(x_2)$ then there exists $x \in X$ so that $f(x) = y$.
2. If $f: [a, b] \to \mathbb{R}$ is continuous, then f has the **intermediate value property**: If $\min\big(f(a), f(b)\big) \leq y \leq \max\big(f(a), f(b)\big)$ then there exists $x \in [a, b]$ so that $f(x) = y$.

X is **pathwise connected** if for any $x_0, x_1 \in X$ there exists a continuous function $f: [0,1] \to X$ so that $f(0) = x_0, f(1) = x_1$. Any pathwise connected set is connected.
Pf. If X is a disjoint union of nonempty open sets, take $x_0 \in U_0, x_1 \in U_1$, let f connect them. Take f^{-1} of U_0, U_1; we get that [0,1] is disconnected, contradiction. (Topo)
Counterexample to converse: Topologist's sine curve $\left\{\left(x, \sin\left(\frac{1}{x}\right)\right)\right\} \cup \{(0, y) | y \in [-1,1]\}$ is

	connected but not pathwise connected.
4-4	Discontinuities One-sided limits • $\lim_{x\to p^+} f(x) = q$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in E$ with $p < x < p + \delta$ we have $\lvert f(x) - q\rvert < \varepsilon$. • $\lim_{x\to p^-} f(x) = q$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in E$ with $p - \delta < x < p$ we have $\lvert f(x) - q\rvert < \varepsilon$. Discontinuity of the first kind: (a) Jump discontinuity: $f(x+) \neq f(x-)$ (b) Removable discontinuity: $f(x+) = f(x-) \neq f(x)$ The function can be redefined at x to make it continuous. Discontinuity of the second kind: Everything else. Ex. 1. Dirichlet's function $$f(x) = \begin{cases} 0, x \notin \mathbb{Q} \\ 1, x \in \mathbb{Q} \end{cases}$$ 2. Riemann's function is continuous at irrational points, and has removable discontinuities at rational points. ($\mathrm{denom}(x)$ is the denominator of x in lowest terms) $$R(x) = \begin{cases} 0, x \notin \mathbb{Q} \\ \frac{1}{\mathrm{denom}(x)}, x \in \mathbb{Q} \end{cases}$$ 3. $y = \sin\left(\frac{1}{x}\right)$ has a discontinuity of the second kind at $x = 0$. Monotonic functions Increasing: $x < y \Rightarrow f(x) \le f(y)$ Strictly increasing: $x < y \Rightarrow f(x) < f(y)$ Decreasing: $x < y \Rightarrow f(x) \ge f(y)$ Strictly decreasing: $x < y \Rightarrow f(x) > f(y)$ Monotonic: Increasing or decreasing • If f is increasing then $f(x+)$ and $f(x-)$ exist for all $x \in (a, b)$ and $$f(x-) = \sup_{t<x} f(t) \le f(x) \le \inf_{t>x} f(t) = f(x+)$$ Moreover, for all $x < y$, $f(x+) \le f(y-)$. Reverse inequalities for f decreasing. • The only discontinuities of a monotonous function are jump discontinuities. • The set of discontinuities points of a monotonic function are at most countable. _Pf._ For each discontinuity point x, associate it with a rational number in $\left(\min\left(f(x-), f(x+)\right), \max\left(f(x-), f(x+)\right)\right)$. Given an at most countable set $S \subset (a, b), S = \{x_1, x_2, \ldots\}$, there exists a monotonic function $f: (a, b) \to \mathbb{R}$ such that f has discontinuities exactly at S. A: Take any positive convergent series; define $f(x) = \sum_{n,\, x_n < x} a_n$.

5	Differentiation
5-1	Derivative The **derivative** of $f: [a, b] \to \mathbb{R}$ at x is $$\frac{d}{dx}f(x) = f'(x) = \lim_{t \to x} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$ Left and right-sided derivatives are defined with left and right-sided limits. If f' exists f is **differentiable**. Generalizes to vector-valued functions. If f is differentiable at x then f is continuous at x. *Pf.* Multiply derivative by $t - x \to 0$. Rules: 1. $(f+g)' = f' + g'$ 2. $(fg)' = f'g + fg'$ (Pf. add and subtract $f(x)g(t)$.) 3. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (Prove for $f = 1$ and use product rule.) 4. $(g \circ f)' = g'(f(x))f'(x)$ $$(x^n)' = nx^{n-1}$$ Tells us how to differentiate polynomials and rational functions. A **local maximum (minimum)** of $f: X \to \mathbb{R}$ is a point $p \in X$ such that there exists $\delta > 0$ such that $f(q) \le f(p)$ ($f(q) \ge f(p)$) for all $q \in X$ with $d(p, q) < \delta$. If f is defined on $[a, b]$ and has a local maximum or minimum at $x \in (a, b)$, and f' exists, then $f'(x) = 0$. Mean Value Theorem: If f, g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there exists a point $x \in (a, b)$ such that $$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$ In particular, there exists a point $x \in (a, b)$ at which $$f(b) - f(a) = (b - a)f'(x).$$ When $f(a) = f(b)$ this is called Rolle's Theorem. *Pf.* Let $h(x) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t)$. Then $h(a) = h(b) = 0$, need to find $x \in (a, b)$ so that $h'(x) = 0$. Take the point x where h attains maximum or minimum. f is increasing if $f' \ge 0$, constant if $f' = 0$, and decreasing if $f' \le 0$. If f is differentiable on $[a, b]$ then f' satisfies the Intermediate Value Theorem, and cannot have any simple discontinuities. L'Hospital's Rule: Suppose f, g are real and differentiable in (a, b), $g'(x) \ne 0$ for all $x \in (a, b)$, and one of the following holds: 1. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ 2. $\lim_{x \to a} g(x) = \pm\infty$ Then

	$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$ if the latter limit is defined. (This rule may need to be used multiple times, and is true for b.) Higher derivatives: $f^{(n+1)}=(f^{(n)})'$ $C^n[a,b]$ denotes the set of functions $f:[a,b]\to\mathbb{R}$ (or $\mathbb{C}$) with continuous nth derivatives. $C^\infty[a,b]$ denotes the set of functions with derivatives of all orders. For vector-valued functions: Suppose $f:[a,b]\to\mathbb{R}^n$ is continuous and differentiable in (a,b). There exists $x\in(a,b)$ so that $$\lvert f(b)-f(a)\rvert\le(b-a)\lvert f'(x)\rvert.$$ *Pf.* Project onto line connecting $f(a)$ with $f(b)$.
5-2	## Taylor and Power Series Power series (and Laurent series) are continuous in the open ball of convergence. *Pf.* If $r<R$ where R is the radius of convergence, then f is uniformly continuous on $(-r,r)$. Factor out $z-w$ from each term z^n-w^n in $f(z)-f(w)$ and use Triangle Inequality and Root Test. If f has derivatives of all orders at α, the **Taylor series** of f around α is $$P(t)=\sum_{k=1}^{\infty}\frac{f^k(\alpha)}{k!}(t-\alpha)^k.$$ Taylor's Theorem: Suppose f is a real function on $[a,b]$, n is a positive integer, f^{n-1} is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t\in(a,b)$. Let α,β be distinct points of $[a,b]$, and let $$P(t)=\sum_{k=1}^{n-1}\frac{f^k(\alpha)}{k!}(t-\alpha)^k.$$ Then there exists a point x between α and β such that $$f(\beta)=P(b)+\frac{f^n(x)}{n!}(\beta-\alpha)^n.$$ *Pf.* Let $M=\frac{f(\beta)-P(\beta)}{(\beta-\alpha)^n}$, so $f(\beta)=P(\beta)+M(\beta-\alpha)^n$. Let $g(t)=f(t)-P(t)-M(t-\alpha)^n$. Then $g^{(n)}(t)=f^{(n)}(t)-n!\,M$. Need $x\in(a,b)$ so that $g^{(n)}(x)=0$. $g^{(k)}(\alpha)=0$ for $0\le x<n$ and $g(\beta)=0$. By induction and the Mean Value Theorem, there exists $x=x_n$ such that $g^n(x)=0$. *Remarks:* For $n=1$ this is the Mean Value Theorem. Useful when there is a convenient upper bound for $f^{(n)}(x)$ on (a,b).

6 Riemann Integration

6-1 Riemann-Stiltjes Integral

A partition P of $[a, b]$ is a finite collection of points $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Define $\Delta x_i = x_i - x_{i-1}$.
Let f be bounded on $[a, b]$, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$.

1. The lower integral sum is $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$. The lower integral is
$$\underline{\int_a^b} f(x)dx = \sup_P L(P, f).$$
2. The upper integral sum is $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$. The upper integral is
$$\overline{\int_a^b} f(x)dx = \sup_P L(P, f).$$

(Exist when f bounded.)
f is **Riemann integrable** on $[a, b]$ if the lower and upper integral sums are equal. Then
$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$
(If $a = b$ the integral is 0.)

More general context: Let α be a monotonically increasing function on $[a, b]$, and let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Define sums and integrals ($d\alpha$) similarly with Δx_i replaced by $\Delta\alpha_i$. The integral is called the **Riemann-Stieltjes integral**. The set of Riemann-Stieltjes integrable functions with respect to α is denoted $R(\alpha)$.
Useful in probability- random variables. If α is the distribution function,$\int_a^b f d\alpha$ is the expected value of $f(X)$.

The integral of a vector (or complex) valued function is taken componentwise (real and imaginary parts separately).

Integrability

A refinement of a partition obtained from adding a set of division points. Any two partitions have a common refinement.
Preliminary Results:

- If P^* is a refinement of P then
  - $L(P, f, \alpha) \leq L(P^*, f, \alpha)$
  - $U(P, f, \alpha) \geq U(P^*, f, \alpha)$
- $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$.
- $f \in R(\alpha)$ iff for every $\varepsilon > 0$ there exists a partition P such that (*) $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.
  - If (*) holds, then it holds for any refinement of P.
  - If (*) holds for $P = \{x_0, x_1, \dots, x_n\}$ and $s_i, t_i \in [x_{i-1}, x_1]$ then $\sum_{i=1}^{n} |f(s_i) - f t i \Delta \alpha i < \varepsilon$.
  - If (*) holds and $f \in R(\alpha)$ then $\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon$.

Main Results:

- Let f be continuous on $[a,b]$. Then f is Riemann-Stieltjes integrable for any α. Proof:
    - f is uniformly continuous. Given ε' choose δ, take partition so that all intervals are shorter than δ. Then (*) holds (choosing ε' depending on ε small enough).
- If f is monotonic and α is continuous then $f \in R(\alpha)$. Proof:
    - By IVT and continuity of α, we can choose a partition so that $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$. Choose n large enough to make (*) hold.
- If f is bounded with only finitely many discontinuity points and α is continuous at all these points, then $f \in R(\alpha)$.
    - Take very small intervals around discontinuity points: Surround the discontinuity points by nonoverlapping intervals $[u_i, v_i]$ where α changes by less than ε_i, where $\sum \varepsilon_i = \varepsilon$. Delete the intervals (and endpoints if they are discontinuity points). f is uniformly continuous on the resulting set (union of closed intervals), take a partition with intervals of length at most $\delta(\varepsilon)$, including $[u_i, v_i]$. Upper bound depends on ε and $M = \sup_{[a,b]} |f|$ (for the intervals $[u_i, v_i]$), and can be made small. (Sum consists of two parts.)
    - Counterexample when α not continuous: $\theta(x) = \begin{cases} 0, x \le 0 \\ 1, x > 0 \end{cases}$. $\int_{-1}^{1} \theta d\theta$ does not exist.
- A set $S \subseteq \mathbb{R}$ has **measure 0** if for all $\varepsilon > 0$ there exists a countable collection of open intervals $(a_i, b_i), i = 1,2,\ldots$ such that $S \subseteq \bigcup_i (a_i, b_i)$ and $\sum_i b_i - a_i < \varepsilon$. f is Riemann integrable on $[a,b]$ iff the set of discontinuity points of f has measure 0.
    - Let B_j be intervals of total length at most $\frac{\varepsilon}{4M}$ covering discontinuity points. Let V be the union of "bad balls," those where $\sup_{B_j} f - \inf_{B_j} f > \frac{\varepsilon}{2M}$.
    - Lemma: There exists $\delta > 0$ such that if $s < t$, $t - s < \delta$, $\sup_{[s,t]} f - \inf_{[s,t]} f > \frac{\varepsilon}{2M}$, then $[s,t] \subseteq V$, i.e. any small interval with large variation must be contained in V. Proof: Else take a sequence $z_i \in [x_i, y_i] \cap V^c$ that violate the lemma, for $\delta_i \to 0$. By sequential compactness, take a simultaneously convergent subseqence; it must be in V^c since V^c is closed, but must also be in V.
    - Break Riemann sum into two parts: the intervals in V (bounded by $\underbrace{2M}_{\text{max variation}} \underbrace{\left(\frac{\varepsilon}{4M}\right)}_{\text{max total length}}$) and others (bounded by $\underbrace{(b-a)}_{\text{max total length}} \underbrace{\left(\frac{\varepsilon}{2M}\right)}_{\text{max variation}}$).
    - Any subset of a set of measure 0 has measure 0.
    - A closed interval $[a,b]$ doesn't have measure 0. (Use compactness) Sets of measure 0 can't contain an interval.
    - Any countable set (ex. $\mathbb{Q}$) has measure 0. (Choose sequence of lengths to make sum converge to arbitrarily small number.)
    - Cantor set has measure 0.
    - A countable union of sets of measure 0 has measure 0.
    - <u>Baire Category Theorem:</u> $\mathbb{R}$ is not a countable union of nowhere dense sets. (A set S is nowhere dense if $\bar{S}$ does not contain an interval.) I.e. intervals are of the second category.
    - Nowhere dense set of measure >0: Like Cantor set but remove intervals whose sum of lengths is <1.
- Let $f \in R(\alpha)$ on $[a,b]$, $m \le f \le M$, and let $\phi: [m, M] \to \mathbb{R}$ be a continuous function. Then $h = \phi \circ f \in R(\alpha)$ on $[a,b]$.
    - ϕ is uniformly continuous, so can choose δ so $|y - z| < \delta \Rightarrow |\phi(y) - \phi(z)| <$

	ε. Choose P so $U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$. Let $M_i = \sup_{[x_{i-1},x_i]} h, m_i = \inf_{[x_{i-1},x_i]} h$. Divide indices into 2 classes. ▪ $M_i - m_i < \delta$: Bound this part of this sum by $\varepsilon\big(\alpha(b) - \alpha(a)\big) \to 0$. ▪ $M_i - m_i \geq \delta$: Bound this part by $2\big(\sup_{[m,M]} f\big)\sum \Delta\alpha_i \leq 2K\varepsilon \to 0$.
6-3	Properties 1. Linearity: $\int_a^b (cf_1 + f_2)d\alpha = c\int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$. 2. $f_1 \leq f_2 \Rightarrow \int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$. 3. If $a < c < b$ then $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$. 4. $\lvert f(x)\rvert \leq M \Rightarrow \left\lvert\int_a^b f d\alpha\right\rvert \leq M\big(\alpha(b) - \alpha(a)\big)$. 5. For $\int_a^b f d(c\alpha_1 + \alpha_2) = c\int_a^b f d\alpha_1 + \int_b^c f d\alpha_2$. More integrability: (Use composition theorem to prove.) 1. If $f, g \in R(\alpha)$ then $fg \in R(a)$. 2. If $f \in R(\alpha)$ then $\lvert f\rvert \in R(\alpha)$ and $\left\lvert\int_a^b f d\alpha\right\rvert \leq \int_a^b \lvert f\rvert d\alpha$. *Ex.* Let $I(x) = \begin{cases}1, x \geq 0\\ 0, x < 0\end{cases}$. $f \in R(I)$ iff f is right continuous at 0. Then $\int_a^b f dI = f(0)$. Let $\sum_i^\infty c_n$ be a convergent nonnegative series. Let $\alpha(x) = \sum_{n=1}^\infty c_n I(x - s_n)$. For any f continuous, $\int_a^b f d\alpha = \sum_{n=1}^\infty c_n f(s_n)$. Assume α is Riemann integrable on $[a,b]$ and f is bounded on $[a,b]$. Then $f \in R(\alpha)$ iff $f\alpha' \in R$, and if so, $\int_a^b f d\alpha = \int_a^b f\alpha' dx$. "$d\alpha = \alpha' dx$." *Pf.* By the Mean Value Theorem, there exists $t_i \in (x_{i-1}, x_i)$ such that $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$. The upper (lower) sums of the two integrals can be made arbitrarily close; the upper and lower integrals are equal (use refinement). Change of variable: Let $\phi: [A,B] \to [a,b]$ be a strictly increasing continuous function. If $f \in R(\alpha)$ on $[a,b]$ then $f \circ \phi \in R(\alpha \circ \phi)$ and $\int_a^b f d\alpha = \int_A^B f \circ \phi \, d(\alpha \circ \phi)$. *Pf.* A partition of $[a,b]$ induces a partition of $[A,B]$. *Cor.* If ϕ is differentiable and $f \in R$ in $[a,b]$, then $\int_a^b f(x)dx = \int_A^B f\big(\phi(x)\big)\phi'(y)dy$. (If ϕ is strictly monotone then $\int_a^b f(x)dx = \int_A^B f\big(\phi(y)\big)\lvert\phi'(y)\rvert dy$.) Integration and differentiation. Fundamental Theorem of Calculus: Let $f \in R$ on $[a,b]$. For $a \leq x \leq b$ define $F(x) = \int_a^b f(t)dt$. Then F is continuous on $[a,b]$ and it f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$. *Pf.* F is Lipschitz with constant $\sup\lvert f\rvert$ so F is uniformly continuous. Using continuity of f, choose δ from ε; using integral bounds the difference $\left\lvert\frac{F(t)-F(s)}{t-s} - f(x_0)\right\rvert$ is at most ε. Take limit. Integration by Parts: If u, v are differentiable on $[a,b]$, then

	$$\int_a^b uv'dx = uv\|_a^b - \int_a^b vu'dx.$$ <u>*Pf.*</u> $(uv)' = u'v + vu'$ is integrable. Assuming the integrals are defined, for $f: [a,b] \to \mathbb{R}^n$ (or $\mathbb{C}$), $\left\|\int_a^b f\, d\alpha\right\| \le \int_a^b \|f\|\, d\alpha$. (Use Cauchy-Schwarz.)
	## Rectifiable Curves A **curve** in $\mathbb{R}^n$ is a continuous function $\gamma: [a,b] \to \mathbb{R}^n$. If γ is injective it is an **arc**; if $\gamma(a) = \gamma(b)$ it is closed. Let $\Lambda(P,\gamma) = \sum_{i=1}^n \|\gamma(x_i) - \gamma(x_{i-1})\|$ when $P = \{a = x_0 < \cdots < x_n = b\}$. Define $$\Lambda(\gamma) = \sup_P \Lambda(P,\gamma).$$ The curve is rectifiable if $\Lambda(\gamma)$ is finite. *Ex.* Nonrectifiable curve- Koch snowflake. If γ' is continuous on $[a,b]$, then γ is rectifiable, and $\Lambda(\gamma) = \int_a^b \|\gamma'(t)\|dt$. <u>*Pf.*</u> By FTC, $\|\gamma(x_i) - \gamma(x_{i-1})\| \le \int_{x_{i-1}}^{x_i} \|\gamma'(t)\|\, dt$. Summing, γ is rectifiable. Using uniform continuity of γ', take a partition with distances less than $\delta(\varepsilon)$; bound the error by $2\varepsilon(b-a)$. (There are 2 parts to the error, sum of $\left\|\int_{x_{i-1}}^{x_i} \gamma'(x_i) - \gamma'(t)dt\right\| + \left\|\int_{x_{i-1}}^{x_i} \gamma'(t)dt\right\|$.

7	Sequences of Functions
7-1	Uniform Convergence

A sequence of functions f_n converges to f ($f_n \to f$) if $\lim_{n\to\infty} f_n(x) = f(x)$ for all x. In general, pointwise convergence does not preserve limits (continuity), derivatives, or integrals. Convergence for series of functions is defined similarly.

f_n **converges uniformly** to $f: E \to X$ (X a complete metric space) if for every $\varepsilon > 0$ there exists N so that for every $n \geq N$ and $x \in E$, $d\big(f_n(p), f(p)\big) < \varepsilon$.
Cauchy Criterion for Uniform Convergence: $\{f_n\}$ is uniformly convergent iff for all $\varepsilon > 0$ there exists N such that $d\big(f_n(x), f_m(x)\big) < \varepsilon$ for every $m, n \geq N; x \in E$.
Pf. Choose N for $\frac{\varepsilon}{2}$ and use triangle inequality.

Weierstrass M-Test for Uniform Convergence:
Suppose $f_n \to f$ pointwise, and let $M_n = \sup_{x\in E} d(f_n, f)$. Then $f_n \to f$ uniformly iff $M_n \to 0$ as $n \to \infty$.
Cor. Let $f_n: E \to \mathbb{R}$, $|f_n| \leq M_n$. If $\sum_{n=1}^{\infty} M_n$ is convergent then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.
Pf. Use Cauchy criterion on difference of partial sums.

Suppose $\{f_n\}$ converges uniformly to $f: E \to \mathbb{R}$. Let x be a limit point of E (subset of metric space). Then
$$\lim_{t\to x} f(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$
Pf. Let $A_n = \lim_{t\to x} f_n(t)$, $A = \lim_{n\to\infty} f_n(t)$. Choose n and δ so that

1. $|f_n(t) - f(t)| < \frac{\varepsilon}{3}$ for all t (use uniform continuity).
2. $|A_n - A| < \frac{\varepsilon}{3}$.
3. $|f_n(t) - A_n| < \frac{\varepsilon}{3}$ for $0 < d(t,x) < \delta$.

Then $|f(t) - A| < \varepsilon$.
Cor. If $f_n: E \to \mathbb{R}$ is continuous and $f_n \to f$ uniformly on E, then f is continuous on E.

Suppose $\{f_n\}$ is a continuous real-valued functions on a compact set K, $f_{n+1} \leq f_n$, and $f_n \to f$ pointwise. Then $f_n \to f$ uniformly.
Pf. Consider $K_n = \{x \in K | f_n(x) - f(x) < \varepsilon\}$. As closed subsets of K they are compact. By monotonicity, $K_n \supseteq K_{n+1}$. Since $\bigcap_{n\geq 1} K_n = \phi$, one of the sets, and all subsequent sets, are empty.

Let $C(X)$ be the space of bounded continuous functions. For $f, g \in C(X)$ define $\|f\| = \sup|f(x)|$ and $d(f,g) = \|f - g\|$. Then $f_n \to f$ with this metric iff $f_n \to f$ uniformly on X.
$C(X)$ is complete because if $\{f_n\}$ is Cauchy, then it is uniformly convergent. Hence it is continuous (and bounded).
Note continuity is not important. The space of continuous functions $C(K)$ on compact K is a complete metric space.

Integration: Suppose α is a monotonically increasing on $[a,b]$, $f_n \in R(\alpha)$, and $f_n \to f$ uniformly. Then $f \in R(\alpha)$ and $\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f\, d\alpha$.
Pf. Let $\varepsilon = \sup|f_n - f|$. Then $f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n$; both sides go to f; integrate.

$f_n \to f$ uniformly does not imply $f_n' \to f'$.
Suppose f_n are differentiable, $f_n{}'$ converges uniformly, and $f_n(x_0)$ converges for some $x_0 \in [a,b]$. Then f_n is uniformly continuous to some function f, and $\lim_{n\to\infty} f_n'(x) = f'(x)$.
<u>*Pf.*</u> Choose n so that for $m,n \ge N$, $|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$ and $|f_n'(x) - f_m'(x)| < \frac{\varepsilon}{2(b-a)}$. Use MVT for $f_m - f_n$ on t, x to get difference at most $\frac{\varepsilon}{2}$. Using Triangle Inequality and Cauchy criterion, $f_n \to f$ uniformly, f continuous.
Let $\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}$. Then $\phi_n(t)$ is uniformly convergent to $\phi(t)$ when $t \ne x$ so by exchanging limits $\lim_{n\to\infty} f_n'(x) = \lim_{t\to x} \phi(t) = f'(x)$.

Everywhere continuous but nowhere differentiable function:
Weierstrass: $W(x) = \sum_{n=0}^{\infty} 2^{-n} \cos(10^n \pi x)$
Or: Let $s(x) = |x|, |x| \le 1$ have period 2. $f_n(x) = \left(\frac{3}{4}\right)^n s(4^n x)$. $f(x) = \sum_{n=0}^{\infty} f_n(x)$ is nowhere differentiable because of increasing oscillations, but continuous by the M-Test (oscillations on smaller scale). In the difference quotient choose $h_m = \pm\frac{1}{2}4^{-m}$ (direction so that don't hit cusps→s_m linear at this scale scale); f becomes a finite sum but the difference quotient increases as m increases (Δs_m is large; the smaller ones don't cancel out).

Holder ½
(Differentiable functions have Baire category 2 in continuous functions.)

Let X be compact (so continuous functions are bounded) and $C(X)$ be the space of continuous functions $f: X \to \mathbb{R}$ with the metric $d(f,g) = \|f - g\|_\infty = \sup|f(x) - g(x)|$. $C(X)$ is complete: $\mathbb{R}$ is complete so a Cauchy sequence f_n converges uniformly. Since f_n are continuous, their limit is continuous.

Heine-Borel fails: Take f_n to be a function with a spike of height 1 at $\left(\frac{1}{n+1}, \frac{1}{n}\right)$ and 0 elsewhere. $\{f_n\}$ is closed but the functions are all distance 1 from each other.

7-2

Equicontinuity

A family $\mathcal{F}$ of functions $X \to \mathbb{R}$ is **equicontinuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all x, y so that $d(x,y) < \delta$. For a finite collection, this is equivalent to all elements being uniformly continuous.
$\mathcal{F}$ is **uniformly bounded** if there exists M so that $f(x) \le M$ for all $f \in \mathcal{F}$ and $x \in X$.

Suppose X is compact and $\{f_n\}$ is uniformly convergent in $C(X)$. Then $\{f_n\}$ is equicontinuous. (Holds if X is not compact but f_n are uniformly continuous.
<u>*Pf.*</u> $\{f_n\}$ uniformly Cauchy. Choose N for $\frac{\varepsilon}{3}$ and then choose δ for $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$.

<u>Arzela-Ascolli Theorem:</u> If X is compact and $\{f_n\}$ is a pointwise bounded equicontinuous sequence in $C(X)$ then $\{f_n\}$ has a uniformly convergent subsequence.
(Separable X implies existence of pointwise convergent subsequence.)
A closed and bounded equicontinuous family of functions C(X) is compact.
<u>*Pf.*</u>

1. Pointwise bounded implies uniformly bounded: Choose δ for equicontinuity for ε, the

	δ-neighborhoods form an open cover; take a finite subcover $O_\delta(x_i)$ and take $\max(\lvert f(x_i)\rvert) + \varepsilon$. 2. Take a countable dense subset $\{p_1, p_2, \ldots\}$. $\{f_n(p_1)\}_{n=1}^{\infty}$ is bounded so has a convergent subsequence $\{g_{1,n}(p_1)\}_{n=1}^{\infty}$. Given $\{g_{i,n}\}$, take a convergent subsequence $\{g_{i+1,n}(p_{i+1})\}$. $\{g_{i,n}\}$ is row i. Take the diagonal $g_{n,n}$; by $\frac{\varepsilon}{2}$-argument, $\{g_k(p)\}$ converges. 3. For $\varepsilon > 0$, choose $\delta > 0$ for equicontinuity for $g_{n,n}$ for $\frac{\varepsilon}{3}$. $B_\delta(p)$ covers X; take a finite subcover $B_\delta(p_i)$. For each p_i take N_i so $\lvert g_k(p_i) - g_l(p_i)\rvert < \frac{\varepsilon}{3}$ for $k, l \geq N_i$. Take $N = \max N_i$. For this N_i, compare $g_k(x), g_l(x)$ to $g_k(p_i), g_l(p_i)$ to show $\lvert g_k(x) - g_l(x)\rvert < \varepsilon$. 4. A closed, bounded, and equicontinuous family in $C(X)$ is sequentially compact so it is compact. *Cor.* If functions f_n defined on a compact set converge pointwise and are equicontinuous, then they converge uniformly. Application: Show the existence of the solution to a differential equation. Solution to $F(f, f') = 0$ is the minimizer of $G: C(X) \to \mathbb{R}$ given by $\int \lvert F(f, f')\rvert^2 \, dx$. Restricting to a compact set of $C(X)$, if G is continuous there must be a minimum.
7-3	Approximation Theorems An algebra $\mathcal{A}$ of functions is a set of functions closed under addition, multiplication, and scalar multiplication. $\mathcal{A}$ is self-adjoint if $f \in \mathcal{A}$ implies $\bar{f} \in \mathcal{A}$. The **uniform closure** of $\mathcal{A}$ is the set of limits of uniform convergent sequences in $\mathcal{A}$; i.e. the closure of $\mathcal{A}$ in the uniform metric. If $\mathcal{A}$ is its own uniform closure, then $\mathcal{A}$ is uniformly closed. Weierstrass Approximation Theorem: Let $[a, b]$ be a compact interval in $\mathbb{R}$, and let $f: [a, b] \to \mathbb{C}$ be continuous. Then there exists a sequence of polynomials P_n such that $\lVert P_n - f_n\rVert \to 0$ on $[a, b]$. I.e. the uniform closure of the set of polynomials on $[a, b]$ is $C[a, b]$. *Pf.* WLOG $[a, b] = [0,1]$ and $f(0) = f(1) = 0$. Set $u_n(x) = c_n(1 - x^2)^n$. 1. Choose c_n so that $\int_0^1 u_n(x)\, dx = 1$. $c_n \sim \sqrt{\frac{x}{\pi}}$. 2. u_n converges to 0 uniformly on $\{x: \lvert x\rvert > \delta\}$ for $\delta > 0$. The polynomials "squish" to 0 and become higher at 0. Let $\bar{f}(x) = \begin{cases} f(x), 0 \leq x \leq 1 \\ 0, \text{otherwise} \end{cases}$. Let $P_n(x) = \int_{-x}^{1-x} \bar{f}(x + t) u_n(t)\, dt = \int_0^1 f(s) u_n(s - x)\, ds$ (a convolution). Let $u_n(s - x) = \sum_{k=0}^{2n} a_k(s) x^k$. Pick $\delta > 0$ so that $\lvert x - y\rvert < \delta \Rightarrow \lvert \bar{f}(x) - \bar{f}(y)\rvert < \frac{\varepsilon}{2}$. Then $P_n(x) - f(x) = \int_{-1}^{1} [\bar{f}(x + t) - f(x)] u_n(t)\, dt$. Split into $\int_{-1}^{-\delta}, \int_{\delta}^{1}, \int_{-\delta}^{\delta}$. *Cor*. There exists a sequence of polynomials $P_n(x)$ such that $P_n(0) = 0$ and $P_n(x) \to \lvert x\rvert$ uniformly on $[-a, a]$. Stone-Weierstrass Theorem: Let K be a compact metric space, and $\mathcal{A} \subseteq C(K, \mathbb{R})$ ($\mathcal{A}$ is a subalgebra of the set of continuous function from K to $\mathbb{R}$). Suppose that $\mathcal{A}$… • Separates points: for every $x_1, x_2 \in K$ there exist $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$. • Does not vanish at any point: there does not exist x such that $f(x) = 0$ for all $f \in \mathcal{A}$.

Then $\mathcal{A}$ is dense in $C(K)$; i.e. the uniform closure $\mathcal{B}$ of $\mathcal{A}$ is the set of all continuous functions.
If $\mathcal{A}$ is a self-adjoint algebra of complex functions that separates points and does not vanish at any point, then $\mathcal{A}$ is dense in $C(K, \mathbb{C})$.
Pf.

1. For every $x_1, x_2 \in K, c_1, c_2 \in \mathbb{R}$ there exists $f \in \mathcal{A}$ such that $f(x_1) = c_1$, $f(x_2) = c_2$.
2. $f \in \mathcal{B} \Rightarrow |f| \in \mathcal{B}$. Let $a = \sup|f(x)|$. Take $P(y)$ so that $\left|P(y) - |y|\right| < \varepsilon$ on $[-a, a]$. Then $\left|P\big(f(x)\big) - |f(x)|\right| < \varepsilon$ on K.
3. $f, g \in \mathcal{B} \Rightarrow \max(f, g), \min(f, g) \in \mathcal{B}$. $\max(f, g) = \frac{f+g+|f-g|}{2}$.
4. For $f \in C(K)$, there exists $g_x \in \mathcal{B}$ such that $g_x(x) = f(x)$ and $g_x(t) > f(t) - \varepsilon$. From (1) take h_y so that $h_y(x) = f(x)$, $h_y(y) = f(y)$. There exists an open set U_y so that $h_y(t) > f(t) - \varepsilon$. Take a finite subcover $\bigcup U_{y_i}$; take $g_x = \max(h_{y_i})$.
5. For each x there exists V_x containing x so that $g_x(t) < f(t) + \varepsilon$. Take a finite subcover $\bigcup V_{x_i}$ and let $h(x) = \min\big(g_{x_i}\big)$.
6. For complex: Use $\Re(f) = \frac{f+\bar{f}}{2}$.

Corollary: Functions $[0,2\pi) \to \mathbb{R}$ can be uniformly approximated by trigonometric polynomials (linear combinations of $\sin(nx), \cos(nx), 1$). Any complex continuous function on the unit circle can be uniformly approximated by Laurent polynomials.

8 Power Series

8-1 Analytic Functions

A function f on $(-a, a)$ is **analytic** if it is representable as the sum of a convergent power series $\sum_{n=0}^{\infty} c_n x^n$.

A power series with radius of convergence b is uniformly convergent on $[-b, b]$ for all $b < a$ and

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \text{ for all } x \in (-a, a).$$

By induction, $f \in C^{\infty}$; i.e. all derivatives exist.
<u>*Pf.*</u> $\sum_{n=1}^{\infty} n c_n x^{n-1}$ converges uniformly by the Root Test.

An analytic function is determined completely by all its derivatives at 0, in particular by values of f in $(-\varepsilon, \varepsilon)$ for any $\varepsilon > 0$. We can define an **analytic continuation**.

Suppose $\sum_{n=0}^{\infty} c_n$ is a convergent series. Let $f(x) = \sum_{n=0}^{\infty} c_n x^n, -1 < x < 1$. Then $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$.
<u>*Pf.*</u> $|f(x) - s| = |(1-x) \sum_{n=0}^{\infty} (s_n - s) x^n| \to 0$.
<u>*Cor.*</u> Suppose $A = \sum_{n=0}^{\infty} a_n, B = \sum_{n=0}^{\infty} b_n, C = \sum_{n=0}^{\infty} c_n$, $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. Then $C = AB$.
This is true if A or B converges absolutely. Else, let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$, $h(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $f(x) g(x) = h(x)$ for $x < 1$; take $x \to 1$.

Inversion of order of sums: If $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ converges, then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$.
<u>*Pf.*</u> Take $E = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$; let $f_i\left(\frac{1}{n}\right) = \sum_{j=1}^{n} a_{ij}, f_i(0) = \sum_{j=1}^{\infty} a_{ij}$. $f_i \to f$ uniformly on E so we can exchange double sums.

<u>Taylor's Theorem:</u> Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$. If $|a| < R$ then f can be expanded into a power series about $x = a$ which converges for $|x - a| < R - |a|$, and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$.
<u>*Pf.*</u> $f(x) = \sum_{n=0}^{\infty} c_n \big((x-a) + a\big)^n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n \binom{n}{m} (x-a)^m a^{n-m}$. Change order of sum (legal since $|x - a| < R - |a|$ gives absolute convergence by applying Binomial Theorem backwards): $\sum_{m=0}^{\infty} [\sum_{n=0}^{\infty} c_n \binom{n}{m} a^{n-m}] (x-a)^m$ converges since $|a| < R$. The series must be its Taylor series.
Suppose $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n$ converge for $|x| < R$, and let $E = \{x | \sum a_n x^n = \sum b_n x^n\}$. If E is not discrete in $(-R, R)$ (i.e. has a limit point in $\mathbb{R}$) then $a_n = b_n$. Let A be the set of all limit points of E in $(-R, R)$ and $B = A^c$. A is closed so B is open. However A is open: Expanding $f(x) = \sum_{n=0}^{\infty} (a_n - b_n) x^n$ near $x_0 \in A$, we get either $f(x) \sim a(x - x_0)^n$ as $x \to x_0$ for some n $\Rightarrow f(x) \neq 0$ in a neighborhood of $f(x) \Rightarrow x_0 \notin E$, or $f(x) = 0$ in a neighborhood of $x_0 \Rightarrow x_0$ is internal point of E.
Since A is both open and closed, either $A = (-R, R) \Rightarrow f(x) = 0$ or $A = \phi \Rightarrow E$ discrete.