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A **topology** on a set $X$ is a collection $\mathcal{T}$ of subsets, called **open sets** satisfying:

1. $\emptyset, X \in \mathcal{T}$
2. The union of an arbitrary collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of a finite number of sets in $\mathcal{T}$ is in $\mathcal{T}$.

$(X, \mathcal{T})$ is called a topological space.

A is **closed** if $X - A$ is open.

1. $X, \emptyset$ are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

If $\mathcal{T}, \mathcal{T}'$ are two topologies for $X$, and

1. $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}$ is coarser than $\mathcal{T}'$.
2. $\mathcal{T} \supseteq \mathcal{T}'$, then $\mathcal{T}$ is finer than $\mathcal{T}'$.
3. Either way, $\mathcal{T}, \mathcal{T}'$ are comparable.

Examples: (Collection of open sets)

1. Discrete topology: Collection of all subsets of $X$
2. Trivial topology: Collection only containing $X, \emptyset$.
3. Finite complement topology: Collection of all subsets $U$ with $X - U$ finite, plus $\emptyset$.

A **base** for a topology on $X$ is a collection $\mathcal{B}$ of subsets, called base elements, of $X$ such that any of the following equivalent conditions is satisfied.

1. Both the following are true.
   a. For each $x \in X$, there is at least one base element containing $x$.
   b. If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $x \in B_3 \subseteq B_1 \cap B_2$ for some $B_3 \in \mathcal{B}$.
2. For each open set $U$ of $X$ and each $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
3. $\mathcal{T}$ (the open sets) is the collection of all unions of elements of $\mathcal{B}$. (Thus the base determines the topology.)

Let $\mathcal{B}, \mathcal{B}'$ be bases for $\mathcal{T}, \mathcal{T}'$ on $X$. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and each base element $B \in \mathcal{B}$ containing $x$, there is a base element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Examples:

1. The standard topology on the real line is generated by open intervals $(a, b)$.
2. The lower limit topology $\mathbb{R}_l$ is generated by half-open intervals $[a, b)$.

A **subbase** $\mathcal{S}$ for a topology is a collection of subsets of $X$ whose union equals $X$, and where $\mathcal{T}$ consists of all unions of finite intersections of elements in $\mathcal{S}$.

Let $X$ be a totally ordered set, and define:

1. Open interval: $(a, b) = \{x \mid a < x < b\}$
2. Half-open interval: \((a, b] = \{x | a < x \leq b\}, [a, b) = \{x | a \leq x < b\}\)
3. Closed interval: \([a, b] = \{x | a \leq x \leq b\}\)

The order topology on \(X\) is the topology generated by the base containing...
1. Open intervals \((a, b)\)
2. Intervals of the form \([a_0, b), a_0 = \min X, \text{if it exists}\)
3. Intervals of the form \((a, b_0], b_0 = \max X, \text{if it exists}\)

1-3 Limit Points and Convergence

Let \(A\) be a subset of \(X\).
1. The interior \(\text{Int}(A) = A^o\) of \(X\) is the union of all open sets contained in \(A\).
2. The closure \(\bar{A} = [A]\) of \(X\) is the intersection of all closed sets containing \(A\).

If \(A\) is open, \(\text{Int}(A) = A\) and if \(A\) is closed, \(\bar{A} = A\).

Let \(Y\) be a subspace of \(X\). Then the closure of \(A\) in \(Y\) is \(\bar{A} \cap Y\).

A subset \(A \subseteq X\) is dense in \(X\) if \(A^c\) is dense in \(X\). An open set containing \(x\) is a neighborhood of \(x\).
1. \(x \in \bar{A}\) iff any neighborhood of \(x\) intersects \(A\).
2. Given a base for \(X\), \(x \in \bar{A}\) iff every base element \(B\) containing \(x\) intersects \(A\).

\(x\) is a limit point of \(A\) if every neighborhood of \(x\) intersects \(A\) in some point other than \(x\). Let \(A'\) be the set of all limit points; then \(\bar{A} = A \cup A'\). \(A\) is closed iff it contains all of its limit points.

\(\{x_n\}_{n=1}^\infty\) converges to \(x\) if for any neighborhood \(U\) of \(x\) there exists \(N \in \mathbb{N}\) such that \(x_n \in U\) for \(n \geq N\). Note that a sequence may converge to more than one point, since one-point sets may not be closed.

1-4 Product and Subspace Topology

For topological spaces \(X, Y\), the product topology on \(X \times Y\) is the topology with base consisting of all the subsets of the form \(U \times V\), where \(U, V\) are open subsets of \(X, Y\), respectively.

If \(\mathcal{B}\) is a base for \(X\) and \(\mathcal{C}\) a base for \(Y\), then \(\mathcal{D} = \{B \times C | B \in \mathcal{B}, C \in \mathcal{C}\}\) is a base for the topology for \(X \times Y\).

\(S = \{\pi_1^{-1}(U) | U \text{ open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ open in } Y\}\) is a subbase for the product topology.

Ex. \(\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}\) is the standard topology for \(\mathbb{R}^2\).

If \(Y \subseteq X\), the collection \(\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}\) is the subspace topology. If \(A \subseteq X, B \subseteq Y\), then the product topology \(A \times B\) is the subspace topology of \(A \times B\) in \(X \times Y\).

\(A\) is closed in \(Y\) iff it is the intersection of a closed set in \(X\) with \(Y\). If \(Y\) is closed in \(X\), then \(A \subseteq Y\) is closed in \(Y\) iff it is closed in \(X\).

Let \(X\) be a totally ordered set and \(Y\) be a subset. If \(Y\) is convex (i.e. \(a, b \in Y\) imply \((a, b) \in Y\)) then the order topology on \(Y\) is the same as the subspace topology.
### 1-5 Infinite Product Topology, Tychonoff Theorem

Let \( \{X_\alpha\}_{\alpha \in I} \) be an indexed family of sets. Consider their Cartesian product \( P = \prod_{\alpha \in I} X_\alpha \). (If all the sets are the same, this is denoted \( X^I \).)

1. **The box topology** is generated by the base of sets \( \prod_{\alpha \in I} U_\alpha \) where \( U_\alpha \) is open in \( X_\alpha \).

2. **The product topology** generated by the subbase

   \[ S = \bigcup_{\beta \in J} S_\beta \]

   where \( S_\beta = \{ \pi^{-1}_\beta(U_\beta) \mid U_\beta \text{ open in } X_\beta \} \) and \( \pi_\beta \) is the projection map \( P \to X_\beta \). In other words, it is generated by the base of sets \( \prod_{\alpha \in I} U_\alpha \) where \( U_\alpha \) is open in \( X_\alpha \) and \( U_\alpha \neq X_\alpha \) only for a finite number of \( \alpha \).

For finite products the two topologies are the same. The box topology is finer than the product topology.

In either topology, \( \prod A_\alpha = \overline{\prod A_\alpha} \).

**Tychonoff Theorem:** An arbitrary product of compact spaces \( X = \prod_{\alpha \in I} X_\alpha \) is compact in the product topology.

**Pf.** Let \( \mathcal{A} \) be a collection of subsets of \( X \) having the finite intersection property; we need to show \( \bigcap \mathcal{A} \neq \emptyset \). By Zorn’s Lemma there is a maximal collection \( \mathcal{D} \) of subsets of \( X \) with the finite intersection property and which contains \( \mathcal{A} \). By compactness of \( X_\alpha \), choose \( x_\alpha \in \bigcap_{\mathcal{D} \subseteq \mathcal{D}} \pi_\alpha(D) \); let \( x = (x_\alpha)_{\alpha \in I} \). Any subbase element containing \( x \) intersects every element of \( \mathcal{D} \). By maximality of \( \mathcal{D} \), any finite intersection of elements of \( \mathcal{D} \), and any subset that intersects every element of \( \mathcal{D} \), is an element of \( \mathcal{D} \). Then every subbase element \( \pi^{-1}_\beta(U_\beta) \) containing \( x \), and every base element containing \( x \), belongs to \( \mathcal{D} \). Then every base element containing \( x \) intersects every element of \( \mathcal{D} \), i.e. any neighborhood of \( x \) intersects \( \bigcap \mathcal{D} \). Hence \( x \in \bigcap_{\mathcal{D} \subseteq \mathcal{D}} D \subseteq \bigcap \mathcal{A} \).

(The proof for finite products does not need Axiom of Choice.)

### 1-6 Axioms of Separation

**Axioms of separation:**

1. For every pair \( x, y \in X \) of distinct points, there exists a neighborhood of \( x \) not containing \( y \). Equivalently, every finite point set is closed.

2. For every pair \( x, y \in X \) of distinct points, there exist neighborhoods \( O_x, O_y \) of \( x \) and \( y \) that are disjoint.

A topological space \( X \) is a **T_1 space** if it satisfies the first axiom of separation and a **Hausdorff (T_2) space** if it satisfies the second axiom of separation.

In a **regular (T_3) space**, 1-point sets are closed and for \( x \in X \) and closed \( B \subseteq X \) not containing \( x \), there exist disjoint open sets containing \( x \) and \( B \).

In a **completely regular (T_{3.5}) space**, 1-point sets are closed and for each \( x \in X \) and closed \( B \subseteq X \) not containing \( x \), there exists a continuous function \( f : X \to [0,1] \) such that \( f(x) = 0 \) and \( f(A) = \{0\} \).

In a **normal (T_4) space**, for every pair of disjoint closed sets \( F_1, F_2 \subseteq X \) there exists disjoint open sets \( O_1 \supseteq F_1 \) and \( O_2 \supseteq F_2 \).

In a **completely normal (T_5) space**, every subspace is normal.

A space is completely regular iff it is the subspace of a normal space.

If \( X \) is a **T_1** space, then \( x \) is a limit point of \( A \subseteq X \) iff every neighborhood of \( x \) contains
In a Hausdorff space, every sequence of points in \( X \) converge to at most 1 point (called the limit).

The following are Hausdorff:
- 1. Simply ordered sets in the order topology
- 2. Product of Hausdorff spaces (with either the box or product topologies)
- 3. Subspace of Hausdorff spaces

A subspace of a regular space is regular and a product of regular spaces is regular.

### 1-7 Countability Axioms

X has a **countable neighborhood base** at \( x \) if there is a countable collection \( \mathcal{B} \) of neighborhoods of \( x \) such that each neighborhood of \( x \) contains at least one of the elements of \( \mathcal{B} \).

**Countability axioms:**
- 1. \( X \) has a countable neighborhood base at each point. (First-countable)
- 2. \( X \) has a countable base for its topology. (Second-countable)

\( X \) is a **Lindelöf space** if every open cover of \( X \) contains a countable subcover.

\( X \) is **separable** if it has a countable dense subset.

A subspace of a 1st (2nd)-countable space is 1st (2nd)-countable, and a countable product of 2nd-countable space is 2nd-countable.

If \( X \) is 2nd-countable (has a countable base) then it is Lindelöf and separable. All three conditions are equivalent if \( X \) is a metric space.

### 1-8 Continuous Functions

Let \( X, Y \) be topological spaces. \( f: X \to Y \) is **continuous** if for each open subset \( V \) of \( Y \), \( f^{-1}(V) \) is an open subset of \( X \). It suffices to show that \( f^{-1}(V) \) is open for every subbase element. The following are equivalent:
- 1. \( f \) is continuous.
- 2. For every subset \( A \) of \( X \), \( f(A) \subseteq f(\overline{A}) \).
- 3. For every closed set \( B \subseteq Y \), \( f^{-1}(B) \) is closed in \( X \).
- 4. For each \( x \in X \) and each neighborhood \( V \) of \( f(x) \), there is a neighborhood \( U \) of \( x \) such that \( f(U) \subseteq V \). (\( f \) is continuous at each point \( x \))

If \( f \) is bijective and \( f, f^{-1} \) are both continuous, then \( f \) is a **homeomorphism**. Here \( U \subseteq X \) is open iff \( f(U) \) is open.

If \( f: X \to Y \) is a homeomorphism when the range is restricted to \( f(X) \), then \( f \) is an **imbedding** of \( X \) in \( Y \).

**Constructing continuous functions**
- 1. Constant functions \( f(x) \equiv y_0 \) are continuous.
- 2. If \( A \subseteq X \) the inclusion function \( i: A \to X \) is continuous.
- 3. If \( f: X \to Y \) and \( g: Y \to Z \) are continuous, then so is \( g \circ f \).
- 4. The restriction \( f|_A \) of a continuous function \( f \) to a subspace \( A \subseteq X \) is continuous.
- 5. If \( f: X \to Y \) is continuous, and \( f(X) \subseteq Z \), then \( f: X \to Z \) is continuous. (change range)
- 6. (Local formulation of continuity) \( f: X \to Y \) is continuous if \( X \) can be written as the union of open sets \( U_\alpha \) such that \( f|_{U_\alpha} \) is continuous for each \( \alpha \).
7. (Pasting lemma) Let \( X = A \cup B \) where \( A, B \) are closed in \( X \). Let \( f: A \to Y \) and \( g: B \to Y \) be continuous. If \( f(x) = g(x) \) for every \( x \in A \cap B \) then \( f \) and \( g \) combine to give a continuous function \( h: X \to Y \):

\[
h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}
\]

8. Let \( f: A \to \prod_{\alpha \in J} X_\alpha \) be given by \( f(a) = (f_\alpha(a))_{\alpha \in J} \). Then \( f \) is continuous iff \( f_\alpha: A \to X_\alpha \) is continuous for all \( \alpha \).

1-10 Quotient Topology

A surjective map \( p: X \to Y \) is a quotient map when \( U \subseteq Y \) is open in \( Y \) iff \( p^{-1}(U) \) is open in \( X \). (Equivalently, \( A \subseteq Y \) is closed in \( Y \) iff \( p^{-1}(a) \) is closed in \( X \).) An open map sends open sets to open sets, and a closed map sends closed sets to closed sets.

If \( A \) is a set and \( p: X \to A \) is surjective, then the unique topology on \( A \) such that \( p \) is a quotient map is the quotient topology induced by \( p \). It consists of the subsets \( U \subseteq A \) such that \( p^{-1}(U) \) is open in \( X \).

Let \( X^* \) be a partition of \( X \) into disjoint subsets whose union is \( X \), and let \( p: X \to X^* \) be the map carrying \( x \) to the element of \( X^* \) containing \( x \). Then \( X^* \) with the quotient topology is called the quotient space of \( X \). (\( X^* \) is obtained by identifying equivalent points.)

A subset \( C \) of \( X \) is saturated with respect to \( p \) if \( C \) contains every set \( p^{-1}(\{y\}) \) that it intersects. If \( A \subseteq X \) is saturated, then the restriction \( p|_A: A \to p(A) \) is a quotient map if \( A \) is open or closed, or \( p \) is an open or closed map.

The composition of two quotient maps is a quotient map.

If each class in \( X^* \) is closed, then \( X^* \) is a \( T_1 \) space.

Let \( p: X \to Y \) be a quotient map. Let \( g: X \to Z \) be a map that is constant on each set \( p^{-1}(\{y\}) \). Then \( g \) factors through \( Y \), i.e. there exists a map \( f: Y \to Z \) so that \( f \circ p = g \). \( f \) is continuous iff \( g \) is continuous; \( f \) is a quotient map iff \( g \) is a quotient map.

If \( g \) is surjective and \( Y = \{g^{-1}(\{z\})|z \in Z\} \) is given the quotient topology, then \( f \) is bijective and continuous; \( f \) is a homeomorphism iff \( g \) is a quotient map. If \( Z \) is Hausdorff, then so is \( X^* \).

1-11 Topological Groups

A topological group \( G \) is a group that is also a \( T_1 \) space, such that the maps

- \( G \times G \to G: (x, y) \to xy \)
- \( G \to G: x \to x^{-1} \)

are continuous.

If \( H \) is a subgroup of \( G \) then collection of cosets \( G/H \) can be given the quotient topology.
Classification of Topological Spaces

**Topological space** $X$

- **T<sub>1</sub> space** (1<sup>st</sup> axiom of separation): (H)
  - Every point $x$ has a countable neighborhood base.
- **2<sup>nd</sup> axiom of countability**: (H)
  - Every point has a countable neighborhood base.
- **T<sub>2</sub> (Hausdorff) space** (2<sup>nd</sup> axiom of separation): (H)
  - For every pair of distinct points, there exists a neighborhood of $x$ not containing $y$.

**Countably (limit point) compact**: (3 equivalent definitions)
- Every infinite subset of $X$ has a finite subcover.
- Every countable open cover of $X$ has a finite subcover.
- Every countable centered system of closed subsets has a non-empty intersection.

**Compact**: (2 equivalent definitions)
- Every open cover of $X$ has a finite subcover.
- Every countable centered system of closed subsets has a non-empty intersection.

**Lindelöf**: Every open cover contains a countable subcover.

**Separable**: (H)
- Contains a countable dense subset

**2<sup>nd</sup> axiom of countability**: (H)
- A countable base exists.

**Sequentially compact**: Every sequence of points has a convergent subsequence

**Compactly generated**: If $A \cap C$ is open in $C$ for each compact $C$, then $A$ is open in $X$.

**Locally compact**: For each $x$ there is some compact subspace $C$ of $X$ that contains a neighborhood of $x$.

**Paracompact**: (H-closed subset)
- Every open covering of $X$ has a locally finite open refinement that covers $X$.

**Compactum**: (H-closed subset, I)
- Every open cover of $X$ has a finite subcover.
- Every countable centered system of closed subsets has a non-empty intersection.

**Manifold**: (metrizable)
- Hausdorff, 2<sup>nd</sup>-countable space such that each point has a neighborhood homeomorphic with an open subset of $\mathbb{R}^n$.

**T<sub>3</sub> (Regular) space**: (H)
- 1-point sets are closed and for $x \in X$ and closed $B \subseteq X$ not containing $x$, there exist disjoint open sets containing $x$ and $B$.

**T<sub>3.5</sub> (Completely regular, Tychonoff) space**: (H)
- 1-point sets are closed and for each $x \in X$ and closed $B \subseteq X$ not containing $x$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = \{0\}$.

**T<sub>4</sub> (Normal) space**: (H-closed subset)
- For every pair of disjoint closed sets $F_1, F_2 \subseteq X$ there exists disjoint open sets $O_1 \supseteq F_1$ and $O_2 \supseteq F_2$.

**T<sub>5</sub> (Completely normal) space**: (H-closed subset)
- Every subspace is normal.

**Baire space**: Given any countable collection \{$A_n$\} of closed sets of $X$ with empty interior, their union $\bigcup_n A_n$ has empty interior.

**Locally (path) connected**: (NH)
- For every neighborhood $U$ of an element $x$ there is a (path) connected neighborhood $V$ of $x$ contained in $U$.

**Locally metrizable**: Every point has a neighborhood that is metrizable in the subspace topology.

**Path connected**: (NH)
- Every pair of points in $X$ can be joined by a path in $X$.

**Connected**: (NH)
- No subsets other than $\emptyset$ and $X$ are both open and closed.

**Locally (path) connected**: (NH)
- For every neighborhood $U$ of an element $x$ there is a (path) connected neighborhood $V$ of $x$ contained in $U$.

**Locally metrizable**: Every point has a neighborhood that is metrizable in the subspace topology.

**Path connected**: (NH)
- Every pair of points in $X$ can be joined by a path in $X$.

**Connected**: (NH)
- No subsets other than $\emptyset$ and $X$ are both open and closed.

**Paracompact**: (H-closed subset)
- Every open covering of $X$ has a locally finite open refinement that covers $X$.

**Totally bounded**: Has a finite $\epsilon$-net (can be covered with a finite number of $\epsilon$-balls for any $\epsilon > 0$).

**Complete**: Every Cauchy sequence converges to an element in $X$.

**Regular + 2<sup>nd</sup> countable** = normal

**Urysohn metrization**: Normal + 2<sup>nd</sup> countable = metrizable

**Nagata-Smirnov metrization**: regular + has countably local base = metrizable

**Smirnov metrization**: Locally metrizable + paracompact = metrizable

In a metric space, compact = countably compact = sequentially compact.

**Paracompact + Hausdorff** = normal

**Regular + Lindelöf** = paracompact
## 2-1 Compactness

An **open cover** of a set $E$ in a topology $X$ is a collection $\mathcal{F}$ of open subsets such that $E \subseteq \bigcup_{G \in \mathcal{F}} G$. A subset $K \subseteq X$ is **compact** if every open cover of $K$ contains a finite subcover. $K$ is **sequentially (or countably) compact** if every infinite subset of $K$ has a limit point in $K$.

On subsets:

- Suppose $K \subseteq Y \subseteq X$. Then $K$ is compact relative to $X$ if it is compact relative to $Y$. In other words, compactness is an intrinsic property.
  1. Closed subsets of compact sets are compact.
  2. Compact subsets of Hausdorff spaces are closed. (Pf. If $Y$ is a compact subspace of the Hausdorff space $X$ and $x_0 \notin Y$ there exist disjoint open sets $U$ and $V$ containing $x_0$ and $Y$, respectively.)

The image of a compact space under a continuous map is compact. (Pf. For an open cover of $f(X)$, take the inverse of each subset. They’re open since $f$ is continuous; choose a finite subcover and take the image.)

If $f : X \to Y$ is a bijective continuous function, $X$ is compact, and $Y$ is Hausdorff, then $f$ is a homeomorphism.

A collection $\mathcal{C}$ of subsets of $X$ has the **finite intersection property** (or is centered) if for every finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$, $\bigcap_{i=1}^n C_i$ is nonempty. (Pf. Prove the contrapositive with the complements.)

$X$ is compact iff for every collection $\mathcal{C}$ of closed sets with the finite intersection property, $\bigcap_{C \in \mathcal{C}} C$ is nonempty.

In particular, any nested sequence of closed nonempty sets in a compact set $X$ has a nonempty intersection. For $\mathbb{R}$ this gives the Nested Intervals Theorem.

In a simply ordered set with the least upper bound property, each closed interval is compact. In particular, closed intervals are compact in $\mathbb{R}$. (Pf. Consider the set of points $y > a$ such that $[a,y]$ can be covered by finitely many elements of the open cover $A$.)

**Extreme Value Theorem:** If $X$ is compact, $f$ is compact, and $Y$ has the order topology, $f : X \to Y$ attains its maximum and minimum.

A point $x$ of $X$ is an **isolated point** if $\{x\}$ is open in $X$.

A nonempty compact Hausdorff space with no isolated points is uncountable. Therefore any closed interval in $\mathbb{R}$ is uncountable.

**Pf.** Given a countable sequence of points $\{x_n\}$, build a nested sequence of open sets $V_1 \supseteq V_2 \supseteq \cdots$ such that $x_n \notin \overline{V_n}$. $x \in \bigcap V_n$ (nonempty by compactness) cannot be in $\{x_n\}$.

## 2-2 Limit Point Compactness

$X$ is **limit point compact** if every infinite subset of $X$ has a limit point. $X$ is **sequentially compact** if every sequence of points of $X$ has a convergent subsequence.

Compactness implies limit point compactness.

All three notions of compactness are equivalent for a metric space (see analysis notes).
2-3 Local Compactness and Compactification

X is **locally compact** at x if there is some compact subspace C of X that contains a neighborhood of x. X is said to be locally compact if it is locally compact at every point.

A **compactification** of a space X is a compact Hausdorff space Y containing X such that \( \bar{X} = Y \). Two compactifications \( Y_1, Y_2 \) are equivalent if there is a homeomorphism \( h: Y_1 \to Y_2 \) such that \( h(x) = x \) for every \( x \in X \).

X is locally compact Hausdorff iff it has a **one-point compactification**, i.e. a compactification Y such that \( Y - X \) consists of a single point. Y is unique up to equivalence. **Pf.**

- **Uniqueness:** Let \( h: Y \to Y' \) be the map that is the identity on X and sends \( p \in Y - X \) to \( q \in Y' - X \). Check that it maps an open set to an open set; if \( U \) contains \( p \) note Y-U is closed and hence compact; its image is compact and hence closed.
- **Existence:** Adjoin an object \( \infty \) to X. Let the open sets be
  1. Open sets in X, and
  2. All sets of the form \( Y - C \), \( C \) compact subspace of X.
Check that X does have the subspace topology. If \( \mathcal{A} \) is an open cover of Y, then \( \mathcal{A} \) must contain a set of type 2. Finitely many of the other subsets cover C; add C. Y is compact: just note if \( Y = \infty \) then take a compact set C containing a neighborhood U of x; U and Y-C are disjoint neighborhoods.
- **Converse:** Choose disjoint \( U,V \) containing \( x,p \). \( Y - V \) is compact and contains U in X.

Let X be Hausdorff. X is locally compact iff given \( x \in X \) and a neighborhood \( U \) of x, there is a neighborhood \( V \) of x such that \( \bar{V} \) is compact and \( \bar{V} \subseteq U \).

An open or closed subset of a locally compact Hausdorff set is locally compact.

A space is homeomorphic to an open subspace of a compact Hausdorff space iff it is locally compact Hausdorff.

2-4 Stone-Čech Compactification

Let X be completely regular. There exists an unique compactification (called the **Stone-Čech compactification** \( \beta(X) \)) Y of X such that every bounded continuous map \( f:X \to \mathbb{R} \) extends uniquely to a continuous map of Y into \( \mathbb{R} \).

**Pf.**

1. Let \( \{f_a\}_{a \in J} \) be the collection of all bounded continuous real-valued functions on X; let
   \( l_a = [\inf f_a(X), \sup f_a(X)] \). Define \( h: X \to \prod_{a \in J} l_a \) by \( h(x) = (f_a(x))_{a \in J} \). By Tychonoff, \( \prod_{a \in J} l_a \) is compact. \( h \) is an imbedding by the Imbedding Theorem.
2. The compactification Y of X can be identified with \( \bar{h}(X) \). Let \( H: Y \to \prod_{a \in J} l_a \) be its imbedding into \( \prod_{a \in J} l_a \).
3. Any bounded continuous function \( f_a \) can be extended to \( \pi_a \circ H \).
4. (Uniqueness of extension) Let \( A \subseteq X \), let Z be Hausdorff, and let \( f:A \to Z \) be continuous. There is at most one extension of \( f \) to a continuous function \( \bar{A} \to Z \). **Pf.** Suppose there were two \( g, g' \) with \( g(x) \neq g'(x) \). Take disjoint neighborhoods \( U, U' \) containing \( g(x), g'(x) \) and by continuity take \( O_x \) so \( g(O_x) \subseteq U, g'(O_x) \subseteq U' \). But some point in A is in \( O_x \), contradiction.
5. (Uniqueness of compactification)
   a. Given any continuous map \( f:X \to C \) into compact Hausdorff space C, \( f \)
extends uniquely to a continuous map \( g: \beta(X) \to C \). (\textbf{Pf.} Since \( C \) is completely regular it can be imbedded in \([0,1]^I\).)

b. Suppose \( Y_1, Y_2 \) are compactifications. Using (a) on \( i_2: X \to Y_2 \), there is a continuous map \( f_2: Y_1 \to Y_2 \), and vice versa. The composition is the identity on \( X \); by uniqueness of extension it must be the identity.

### 2-5 Paracompactness

\( X \) is **paracompact** if every open cover of \( X \) has a locally finite open refinement that covers \( X \).

Every paracompact Hausdorff space is normal. (first show it's regular)

Every closed subspace of a paracompact space is paracompact.

Let \( X \) be regular. The following conditions are equivalent: Every open cover of \( X \) has a refinement that is...

1. An countably locally finite open cover of \( X \).
2. A locally finite cover of \( X \).
3. A locally finite closed cover of \( X \).
4. A locally finite open cover of \( X \).

(1)\(\Rightarrow\)(2): Write \( B = \bigcup_n B_n \), where \( B_n \) is locally finite. Define \( S_n(U) = U - \bigcup_{i<n, U_i \in B_i} U_i \) (to make sets in different \( C_n \) disjoint—shrinking trick). Let \( C_n = \{S_n(U)|U \in B_n\} \); take \( C = \bigcup_n C_n \).

(2)\(\Rightarrow\)(3): Let \( \mathcal{A} \) be a open cover. Let \( B = \{U \text{ open}|\bar{U} \in A \text{ for some } A \in \mathcal{A}\} \). By regularity, \( B \) covers \( X \). Refine \( B \) to locally finite \( C \) by (2); take \( D = \{\bar{C} \mid C \in C \} \).

(3)\(\Rightarrow\)(4): Let \( \mathcal{A} \) be open cover. Take locally finite refinement \( B \) that covers \( X \). Introduce an auxiliary locally finite closed cover \( C \) and use it to expand elements of \( B \) into open sets: By local finiteness of \( B \), the collection of all open sets that intersect finitely many elements of \( B \) is an open cover. Use (3) to get a closed locally finite refinement \( C \) covering \( X \). Let \( C(B) = \{C \mid C \in C \text{ and } C \subseteq X - B\} \) and \( E(B) = X - \bigcup_{C \in C(B)} C \). To make \( \{E(B)\} \) a refinement, for each \( B \in B \) choose \( F(B) \in \mathcal{A} \) containing \( B \). Let \( D = \{E(B) \cap F(B) \mid B \in B\} \).

Every metrizable space is paracompact. (Pf. Get a countably locally finite refinement cover and use above.)

Every regular Lindelöf space is paracompact.
### 3 Metric Topologies and Continuous Functions

#### 3-1 Metric Spaces

A set $X$ with a real-valued function (a metric) $d(p, q)$ on pairs of points in $X$ is a **metric space** if:

1. $d(p, q) \geq 0$ with equality iff $p = q$.
2. $d(p, q) = d(q, p)$
3. $d(p, q) \leq d(p, r) + d(r, q)$ (Triangle inequality)

A **metric space** is a topology with base consisting of $\varepsilon$-neighborhoods $N(p, \varepsilon) = \{q \in X | d(p, q) < \varepsilon\}$.

$X$ is **metrizable** if there exists a metric $d$ on $X$ that induces the topology of $X$. $X$ is **topologically complete** if there exists a metric so that it is a complete metric space.

Let $d, d'$ be metrics on $X$ inducing topologies $\mathcal{T}, \mathcal{T}'$. Then $\mathcal{T}$ is finer than $\mathcal{T}'$ iff for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that $N_{d'}(x, \delta) \subseteq N_d(x, \varepsilon)$.

#### 3-2 Metrics for $\mathbb{R}^J$

The metric space $\mathbb{R}^n$, with the Euclidean metric

$$d(x, y) = \left( \sum_{k=1}^{n} |x_k - y_k|^2 \right)^{\frac{1}{2}}$$

or with the square metric

$$d(x, y) = \max |x_k - y_k|$$

induce the product topology.

Let $\tilde{d}(x, y) = \min(|x - y|, 1)$, and define the **uniform metric** $\rho(x, y) = \sup \{\tilde{d}(x_a, y_a) | a \in J\}$ on $\mathbb{R}^J$. The uniform topology is finer than the product topology but coarser than the box topology. (They are all different if $J$ is infinite.)

If $J$ is countable then the metric

$$D(x, y) = \sup \left\{ \frac{\tilde{d}(x_i, y_i)}{i} \right\}$$

induces the product topology on $\mathbb{R}^\omega$. When $J$ is infinite, $\mathbb{R}^J$ is not metrizable when $J$ is uncountable or when the box topology is used.

#### 3-3 Continuous Functions in Metric Spaces

The topological and analytic definitions of continuity are equivalent: let $X, Y$ be metric spaces; continuity of $f: X \to Y$ is equivalent to the requirement that for any $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.

**Sequence lemma**: Let $X$ be a topological space and $A \subseteq X$. If there is a sequence of points of $A$ converging to $x$ then $x \in \bar{A}$; the converse holds if $X$ has a countable basis at $x$ (in particular, it holds when $X$ is metrizable).

Let $f: X \to Y$. If $f$ is continuous, then for every convergent sequence $x_n \to x$ in $X$ the sequence $f(x_n)$ converges to $f(x)$. The converse holds if $X$ has a countable basis at $x$. 
Uniform limit theorem: Let $X$ be a topological space, $Y$ be a metric space, and $f_n: X \to Y$ be a sequence of continuous functions. If $(f_n)$ converges uniformly to $f$, then $f$ is continuous.

3-4 Urysohn Metrization Theorem

Urysohn Lemma: Let $X$ be a normal space and $A, B$ be disjoint closed subsets of $X$. For any $a \leq b$ there exists a continuous map $f: X \to [a, b]$ such that $f(x) = a$ for every $x \in A$ and $f(x) = b$ for every $x \in B$. (Any two closed sets can be separated by a continuous function.)

Proof. Order the rational numbers in $[0,1]$: \( \{a_n\}_{n=0}^\infty = 0, 1, \ldots \) By normality, given any set $S$ and open $T$ such that $\overline{S} \subset T$ there exists an open set $U$ such that $\overline{S} \subset U, \overline{U} \subset T$. Let $U_1 = X - B, A \subset U_0$ and inductively define open sets $U_{a_i}$ such that $U_{p} \subset U_q$ when $p < q$. Set $U_p = \phi$ for $p < 0$ and $U_p = X$ for $p > 1$. Let $f(x) = \inf \{p | x \in U_p\}$. This gives $f: X \to [0,1]$.

Imbedding Theorem: Let $X$ be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $X \to \mathbb{R}$ such that for each point $x_\alpha$ and each neighborhood of $x_\alpha$ there is an index $\alpha$ such that $f_\alpha(x_\alpha) > 0$ and $f(X - U) = \{0\}$. Then the function $F: X \to \mathbb{R}^J$ defined by

\[
F(x) = (f_\alpha(x))_{\alpha \in J}
\]

is an imbedding of $X$ in $\mathbb{R}^J$.

Proof. Need to show the inverse of $F$ is continuous; i.e. if $U$ is open then $F(U)$ is open. Let $z_0 \in F(U)$. Choose $N$ so that $f_N(x_0) > 0, f_N(X - U) = \{0\}$. Let $W = \pi_n^{-1}(\{0, \infty\}) \cap F(X)$; then $z_0 \in W \subseteq F(U)$.

Urysohn Metrization Theorem: Every regular space $X$ with a countable base (i.e. $X$ is 2nd countable) is metrizable.

Proof. There exists a countable collection of continuous functions as in the Imbedding Theorem: Take a countable base $\{B_n\}$; for every pair $(m, n)$ so that $\overline{B_n} \subseteq B_m$, use Urysohn to get continuous $g_{n,m}$ with $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X - B_m) = \{0\}$. Use the Imbedding Theorem.

3-5 Tietze Extension Theorem

Tietze Extension Theorem: Let $X$ be normal and let $A$ be a closed subspace of $X$. Any continuous map $f: A \to Y$ where $Y = [a, b]$ or $\mathbb{R}$ may be extended to a continuous map of all $X$ into $Y$.

Proof. First prove it for $[-1, 1]$. Let $I_1 = [-r, -\frac{1}{3}r], I_2 = [-\frac{1}{3}r, \frac{1}{3}r], I_3 = [\frac{1}{3}r, r]$. By Urysohn Lemma, there is a continuous function $g: X \to [-\frac{1}{3}r, \frac{1}{3}r]$ with $g(x) = -\frac{1}{3}r$ for $x \in f^{-1}(I_1)$ and $g(x) = \frac{1}{3}r$ for $x \in f^{-1}(I_3)$. (g is not too large but approximates f.) Apply this result for $r = 1$ and $f$ to get $g_1$, then inductively for $r = \left(\frac{2}{3}\right)^n$ and $f - g_1 - \cdots - g_n$ to get $g_{n+1}$ (Approximate the error in the previous approximation). Then $\sum_{n=1}^{\infty} g_n(x)$ is the desired function; it is equal to $f$ on $A$ and continuous by the Weierstrass M-test.

For $\mathbb{R}$, consider the homeomorphic space $(-1, 1)$. Define $g$ as above; let $D = g^{-1}((-1)) \cup g^{-1}((1))$. By Urysohn, there is continuous $\phi: X \to [0, 1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. The desired function is $\phi(x)g(x)$. (It’s equal to $g$ on $A$, and doesn’t take the value 1.)
A collection \( \mathcal{A} \) of subsets of \( X \) is **locally finite** in \( X \) if every point of \( X \) has a neighborhood intersecting only finitely many elements of \( \mathcal{A} \).

If \( \mathcal{A} \) is locally finite, then \( \{ \overline{A} \mid A \in \mathcal{A} \} \) is locally finite, and \( \bigcup_{A \in \mathcal{A}} \overline{A} = \bigcup_{A \in \mathcal{A}} \overline{A} \).

\( \mathcal{B} \) is **countably locally finite** if \( \mathcal{B} \) can be written as the countable union of collections \( \mathcal{B}_n \), each of which is locally finite.

Let \( \mathcal{A} \) be a collection of subsets of \( X \). A collection \( \mathcal{B} \) **refines** \( \mathcal{A} \) if for each element \( B \in \mathcal{B} \), there is an element \( A \) of \( \mathcal{A} \) containing \( B \). \( \mathcal{B} \) is open/closed if all sets in \( \mathcal{B} \) are open/closed.

Let \( X \) be a metric space. If \( \mathcal{A} \) is an open cover of \( X \), then there is an open cover \( \mathcal{B} \) of \( X \) refining \( \mathcal{A} \) that is countably locally finite.

**Pf.** Well-order \( \mathcal{A} \). For \( U \subseteq X \) define \( S_n(U) = \{ x \mid N(x, 1/n) \subseteq U \} \) (shrinking by \( 1/n \)). Define \( T_n(U) = S_n(U) - \bigcup_{V \subset U} V \).

(Exclude other sets for local finiteness.) Let \( E_n(U) = \bigcup_{x \in T_n(U)} N(x, \frac{1}{3n}) \) (expand \( T_n(U) \) by \( \frac{1}{3n} \); we have \( E_n(U), E_n(V) \) disjoint when \( U \neq V \); note \( E_n(U) \) is open). Let \( \mathcal{B}_n = \{ E_n(U) \mid U \in \mathcal{A} \} \).

A subset \( A \subseteq X \) is a **\( G_\delta \)-set** if it is the intersection of a countable collection of open subsets of \( X \).

Let \( X \) be a regular space with countably locally finite base. Then \( X \) is normal, and every closed set in \( X \) is a \( G_\delta \)-set in \( X \).

**Pf.**
1. Let \( W \) be open. We show there is a countable collection of open sets \( U_n \) so \( W = \bigcup U_n \). Write \( B = \bigcup B_n \) as a union of locally finite collections. Let \( C_n = \{ B \in \mathcal{B}_n \mid B \subseteq W \} \). Let \( U_n = \bigcup_{B \in C_n} B \).
2. Given closed \( C \), write \( X - C = \bigcup U_n \) by (1). Then \( C = \bigcap_n X - U_n \).
3. \( X \) is normal: Let \( C, D \) be disjoint closed sets. By (1) write \( X - D = \bigcup U_n = \bigcup U_n \) and \( X - C = \bigcup V_n = \bigcup V_n \), then \( U = \bigcup_{n=1}^\infty (U_n - U_{i=1}^n \overline{V_i}) \) and \( V = \bigcup_{n=1}^\infty (V_n - U_{i=1}^n \overline{U_i}) \) are disjoint open sets around \( C, D \).

Let \( X \) be normal and \( A \) a closed \( G_\delta \)-set in \( X \). Then there is a continuous function \( f : [0, 1] \) such that \( f(x) = 0 \) for \( x \in A \) and \( f(x) > 0 \) for \( x \not\in A \).

**Pf.** Write \( A = \bigcup U_n \). By Urysohn Lemma choose \( f_n : X \to [0, 1] \) so \( f_n(x) = 0 \) for \( x \in A \) and \( f(x) > 0 \) for \( x \in X - U_n \). Take \( f(x) = \sum_{n=1}^\infty \frac{f_n(x)}{2^n} \).

**Nagata-Smirnov Metrization Theorem:** \( X \) is metrizable iff \( X \) is regular and has a countably locally finite base \( \mathcal{B} \).

**Pf.**
1. \( X \) is normal and every closed set in \( X \) is a \( G_\delta \)-set.
2. Write \( \mathcal{B} = \bigcup_{B_n} \). Let \( f \) be the set of pairs \( (n, B \in \mathcal{B}_n) \). For each pair choose continuous \( f_{n,B} : X \to [0, 1/n] \) so \( f_{n,B}(x) > 0 \) for \( x \in B \) and \( f_{n,B}(x) = 0 \) for \( x \not\in B \). Then \( \{ f_{n,B} \} \) separates points from closed sets in \( X \).
3. Define \( F : X \to [0, 1]^I \) by \( F(x) = (f_{n,B}(x))_{(n,B) \in I} \). \( F \) is an imbedding in the product topology.
4. To show $F$ is an imbedding in the uniform topology, we need to show $F$ is continuous. Note on $[0,1]^I$, $\rho((x_\alpha),(y_\alpha)) = \sup\{|x_\alpha - y_\alpha|\}$. Take $x_0 \in X$. By local finiteness, choose a neighborhood $U_n$ of $x_0$ intersecting finitely many sets in $\mathcal{B}_n$. Then only finitely many of the $f_{n,B}$ are nonzero. By continuity, choose neighborhood $V_n$ of $x$ so that the nonzero functions vary by less than $\varepsilon$. Choose $N$ so $\frac{1}{N} < \varepsilon$. Let $W = \bigcap_{i=1}^N V_i$. Then for $x \in W$, $|f_{n,B}(x) - f_{n,B}(x_0)| < \varepsilon$ (for $n < N$, this is from $x \in V_n$, for $n \geq N$, this is from $f_{n,B}(x) < \frac{1}{N}$).

5. (Converse) Let $\mathcal{A}_m = \{N(x,1/m) | x \in X\}$. There is an open covering $\mathcal{B}_m$ of $X$ refining $\mathcal{A}_m$ that is countably locally finite. Let $\mathcal{B} = \bigcup_m \mathcal{B}_m$.

3-7 Smirnov Metrization

Smirnov Metrization Theorem: $X$ is metrizable iff it is a locally metrizable paracompact Hausdorff space.

*Proof.* Cover $X$ by metrizable open sets; choose a locally finite open refinement $\mathcal{C}$ that covers $X$. Take metrics $d_C : X \times X \to \mathbb{R}$. Let $\mathcal{A}_m = \{N_c(x,1/m) | x \in C$ and $C \in \mathcal{C}\}$. By paracompactness let $\mathcal{D}_m$ be a locally finite open refinement covering $X$; then $\mathcal{D} = \bigcup_m \mathcal{D}_m$ is a countably locally finite base. By Nagata-Smirnov, $X$ is metrizable.

3-8 Topologies on Function Spaces, Arzela-Ascoli Theorem

$X$ is **compactly generated** if whenever $A \cap C$ is open for every compact subspace $C$, we have that $A$ is open in $X$. If $X$ is locally compact, or $X$ is $1^{st}$-countable, then $X$ is compactly generated.

If $X$ is compactly generated then $f : X \to Y$ is continuous iff for each compact subspace $C$ of $X$, $f|_C$ is continuous.

On $Y^X$ (the set of functions from $X$ to $Y$), the **topology of pointwise convergence** is the topology generated by the subbase $S(x,U) = \{f | f(x) \in U\}$ where $U$ is an open set in $Y$.

- A sequence of functions converges to $f$ in the topology of pointwise convergence iff the sequence converges pointwise (i.e. $f_n(x) \to f(x)$ for each $x$).

For a metric space $Y$, the **topology of compact (or uniform) convergence** on $Y^X$ is the topology generated by the base $B_C(f,\varepsilon) = \{g | \sup\{d(f(x),g(x)) | x \in C\} < \varepsilon\}$.

- A sequence $f_n : X \to Y$ converges to $f$ in this topology iff for each compact subspace $C \subseteq X$, the sequence $f_n|_C$ converges uniformly to $f|_C$.
- The space of continuous functions $C(X,Y)$ is closed in the topology of compact convergence. (Pf. uniform limit theorem.)

A generalization of the topology of compact convergence to arbitrary $X,Y$ is the **compact-open topology** on $C(X,Y)$, generated by the subbase $S(C,U) = \{f | f(C) \subseteq U\}$.

- If $Y$ is a metric space, then the compact-open topology and the topology of compact convergence coincide.

Let $Y$ be a metric space. For the function space $Y^X$, the inclusions of topologies is (pointwise convergence)$\subseteq$(compact convergence)$\subseteq$(uniform)

If $X$ is compact, the right two coincide; if $X$ is discrete, the first two coincide.

Note the compact convergence topology does not depend on the metric of $Y$.

Let $X$ be locally compact Hausdorff, and let $C(X,Y)$ have the compact-open topology. Then the evaluation map $e : X \times C(X,Y) \to Y$ defined by $e(x,f) = f(x)$ is continuous.
Let $Y$ be a metric space, and let $\mathcal{F} \subseteq C(X, Y)$. $\mathcal{F}$ is **equicontinuous** at $x_0$ if given $\varepsilon > 0$ there is a neighborhood $U$ of $x_0$ such that for all $x \in U$ and all $f \in \mathcal{F}$, $d(f(x), f(x_0)) < \varepsilon$. If $\mathcal{F}$ is equicontinuous at every point of $X$, it is said to be equicontinuous.\(^1\)

**Arzela-Ascoli Theorem:** Let $Y$ be a metric space. Give $C(X, Y)$ the topology of compact convergence and let $\mathcal{F}$ be a subset of $C(X, Y)$.

1. If $\mathcal{F}$ is equicontinuous and the set $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ has compact closure for each $a \in X$, then $\mathcal{F}$ is contained in a compact subspace of $C(X, Y)$.
2. The converse holds if $X$ is locally compact Hausdorff.

**Pf.**

1. $\mathcal{G} = \overline{\mathcal{F}}$ is a compact subspace of $(\text{Hausdorff}) Y^X$ under the product (pointwise convergence) topology. Indeed, $\mathcal{G}$ is closed and contained in $\prod_{a \in X} \overline{\mathcal{F}_a}$, compact by Tychonoff.
2. $\mathcal{G}$ is equicontinuous. ($\varepsilon$ argument)
3. The product topology on $Y^X$ and the compact convergence topology on $C(X, Y)$ coincide on $\mathcal{G}$. Suffices to show product topology is finer than the compact convergence topology. Given $B_c(g, \varepsilon)$, cover $C$ by finitely many open sets $U_i$ so that $d(g(x), g(x_i)) < \frac{\varepsilon}{3}$ for each $x \in U_i, g \in \mathcal{G}$. Let $B = \{h \in Y^X \mid \forall i, d(h(x_i), g(x_i)) < \frac{\varepsilon}{3}\}$ (a base element for $Y^X$); then $B \cap \mathcal{G} \subseteq B_c(g, \varepsilon) \cap \mathcal{G}$.
4. (part 2) Let $\mathcal{H}$ be a compact subspace of $C(X, Y)$ containing $\mathcal{F}$. It suffices to show $\mathcal{H}$ is equicontinuous and $\mathcal{H}_a$ is compact for each $a \in X$. $\mathcal{H}_a$ is compact because it is the image of a continuous map from $\mathcal{H}$ (via the evaluation map). The restriction map $r: C(X, Y) \to C(A, Y)$ is continuous; thus $R = \{f|_A \mid f \in \mathcal{H}\}$ is compact. The compact convergence and uniform topologies on $C(A, Y)$ coincide.
   a. A metric space is compact iff it is complete and totally bounded.

Let $X$ be a space and $(Y, d)$ be a metric space. If $\mathcal{F} \subseteq C(X, Y)$ is totally bounded under the uniform metric corresponding to $d$ then $\mathcal{F}$ is equicontinuous under $d$.

By (a), $\mathcal{R}$ is totally bounded in the uniform metric on $C(A, Y)$; by (b), $\mathcal{R}$ is equicontinuous relative to $d$.

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\(^1\) Warning: This is different from the analysis definition.
## Connectedness

### 4-1 Connectedness

A *separation* of $X$ is a pair $U, V$ of disjoint nonempty open subsets of $X$ whose union is $X$. $X$ is *connected* if it satisfies any of the two equivalent conditions:

1. There is no separation of $X$.
2. The only subsets of $X$ that are both empty and closed are $\emptyset$ and $X$.

If $Y$ is a subspace of $X$, a separation of $Y$ is a pair of disjoint nonempty sets $A$ and $B$ whose union is $Y$, neither of which contains a limit point of the other. $Y$ is connected if there is no separation of $Y$ (under this definition).

**Basic results:**

1. If $C$ and $D$ form a separation of $X$, and $Y \subseteq X$ is connected, then $Y$ lies entirely within $C$ or $D$.
2. The union of a collection of connected subspaces of $X$ with a point in common is connected.
3. If $A$ is connected and $A \subseteq B \subseteq \overline{A}$ then $B$ is connected.
4. The image of a connected space under a continuous map is connected.
5. A finite Cartesian product of connected spaces is connected.

Define $x \sim y$ if there is a connected subspace of $X$ containing both $x$ and $y$. The equivalence classes are the *connected components* of $X$.

Equivalently, for $x \in E$, the connected component of $x$ is the union of all connected subsets containing $x$.

The connected components form a partition of $E$, they are all closed sets, and every connected subset of $X$ is entirely within one of them.

A simply ordered set $L$ having more than one element is a *linear continuum* if

1. $L$ has the least upper bound property.
2. If $x < y$ there exists $z$ such that $x < z < y$.

If $L$ is a linear continuum in the order topology, then $L$ is connected, as are intervals and rays in $L$. In particular, $\mathbb{R}$ and its intervals and rays are connected.

**Intermediate Value Theorem:** Let $f: X \to Y$ be a continuous map where $X$ is connected and $Y$ has the order topology. If $a, b \in X$ and $c$ is between $f(a), f(b)$, then there exists a point $c \in X$ with $f(c) = r$.

### 4-2 Path connectedness

A *path* in $X$ from $x$ to $y$ is a continuous map $f: [a, b] \to X$ such that $f(a) = x, f(b) = y$. $X$ is *path connected* if every pair of points in $X$ can be joined by a path in $X$.

Any path connected space is connected, but not vice versa.

*Ex.* Topologist’s sine curve $\{ (x, \sin \left( \frac{1}{x} \right)) \mid 0 < x \leq 1 \}$ is connected but not path connected.

Defining $x \sim y$ if there is a path from $x$ to $y$, the equivalence classes are the *path components* of $X$. Each nonempty path connected subspace is entirely within one path component.
Local connectedness

X is **locally (path) connected** at x if for every neighborhood U of x there is a (path) connected neighborhood V of x contained in U. If X is locally (path) connected at every point, it is simply said to be locally (path) connected.

X is locally (path) connected iff for every open set U of X, each (path) component of U is open in X.

Each path component of X lies in a component of X. If X is locally path connected, then the components and path components coincide.
5 | Manifolds and Dimension

5-1 | Baire Spaces

X is a **Baire space** if any of the two following conditions hold:

1. Given any countable collection \( \{A_n\} \) of closed sets of X with empty interior, their union \( \bigcup_n A_n \) has empty interior.
2. Given any countable collection \( \{U_n\} \) of dense open sets of X, their intersection \( \bigcap_n U_n \) is dense in X.

A subset of a space X is of the **first category** if it is contained in the union of a countable collection of closed sets having empty interior, and of the **second category** otherwise. In a Baire space, every nonempty open set is of the second category.

**Baire Category Theorem:** If X is a compact Hausdorff space or a complete metric space (such as \( \mathbb{R} \)) then X is a Baire space.

*Proof.* Given a countable collection \( \{A_n\} \) of closed sets and a nonempty open set \( U_0 \subseteq X \) we find \( x \in U_0 \) not in any of the \( A_n \). Inductively define \( U_n \): choose a point of \( U_{n-1} \) not in \( A_n \), then choose \( U_n \) to be a neighborhood of this point so that \( U_n \cap A_n = \emptyset, U_n \subseteq U_{n-1} \), and in the metric case, \( \text{diam}(U_n) < \frac{1}{n} \). If compactness is assumed, then \( \bigcap_n U_n \) is nonempty. In the metric case, get a Cauchy sequence.

Any open subspace of a Baire space is a Baire space.

Let Y be a metric space. Let \( f_n : X \to Y \) be a sequence of continuous functions converging pointwise to \( f \). If X is a Baire space the set of points at which \( f \) is continuous is dense in X.

5-2 | Imbeddings of Manifolds

A **m-manifold** is a Hausdorff space with countable basis such that each point \( x \in X \) has a neighborhood homeomorphic with an open subset of \( \mathbb{R}^m \). A 1-manifold is a curve and a 2-manifold is a surface.

For \( \phi : X \to \mathbb{R} \), the **support** of \( \phi \) is defined as \( \text{Supp}(\phi) = \overline{\phi^{-1}(\mathbb{R} - \{0\})} \). Let \( \{U_1, \ldots, U_n\} \) be a finite open covering of the normal space X. An indexed family of continuous functions \( \phi_i : X \to [0,1] \) is said to be a **partition of unity** dominated by \( \{U_i\} \) if

1. \( \text{Supp}(\phi_i) \subseteq U_i \) for each \( i \)
2. \( \sum_{i=1}^n \phi_i(x) = 1 \) for each \( x \)

*Proof of existence.* We can shrink \( \{U_i\} \) to an open covering \( \{V_i\} \) of X such that \( \overline{V_i} \subseteq U_i \), by normality. (Define \( V_i \) inductively.) Shrink \( \{V_i\} \) to \( \{W_i\} \) by the same method. By Urysohn’s Lemma choose \( \psi_i : X \to [0,1] \) so that \( \psi_i(\overline{W_i}) = \{1\} \) and \( \psi_i(X - V_i) = \{0\} \). Scale to get \( \phi_i(x) = \frac{\psi_i(x)}{\sum_{i=1}^n \psi_i(x)} \).

If X is a compact m-manifold, then X can be imbedded in \( \mathbb{R}^N \) for some \( N \).

*Proof.* Cover X by finitely many open sets \( \{U_i\} \), where \( U_i \) can be imbedded in \( \mathbb{R}^m \) via \( g_i \). Let \( \phi_1, \ldots, \phi_n \) be a partition of unity dominated by \( U_i \). Let

\[
h_i(x) = \begin{cases} \phi_i(x)g_i(x) & \text{for } x \in U_i \\ 0 & \text{for } x \in X - A_i \end{cases}
\]

Define \( F : X \to ((\mathbb{R})^n \times (\mathbb{R}^m)^n) \) by \( F(x) = (\phi_1(x), \ldots, \phi_n(x), h_1(x), \ldots, h_n(x)) \). \( F \) is injective (\( g_i \) is injective on \( U_i \); for each \( x \) some \( \phi_i(x) \) is positive) so this works.
Dimension Theory

A collection of subsets of $X$ has order $n$ if some point of $X$ lies in $n$ elements of $A$ and no point of $X$ lies in more than $n$ elements of $A$.

$X$ is finite dimensional if there is some integer $m$ such that for every open cover $A$ of $X$, there is an open cover $B$ of $X$ refining $A$ with order at most $m + 1$. The topological dimension $\dim(X)$ of $X$ is the smallest value of $m$ for which this statement holds.

If $Y$ is a closed subspace of $X$ then $\dim(Y) \leq \dim(X)$.

If $X = Y \cup Z$ where $Y, Z$ are closed, then $\dim(X) = \max\{\dim(Y), \dim(Z)\}$.

**Proof.**

1. If $A$ is an open cover, there is an open cover refining $A$ and has order at most $m + 1 = \max\{\dim(Y), \dim(Z)\} + 1$ at points of $Y$. Consider $\{A \cap Y | A \in A\}$; take a open cover refinement $B$. For $B \in B$, choose open $U_B$ so $U_B \cap Y = B$ and choose $A_B \in A$ so $B \subseteq A_B$; take the cover $C = \{U_B \cap A_B\}$.

2. Let $C'$ be a refinement of $C$ with order at most $m + 1$ at points of $Z$. Define $f : C' \to C$ so that $C \subseteq f(C) \subseteq B$. Let $D(B) = \bigcup_{C \in C', f(C) = B} C$. Take $D = \{D(B)\}$.

Every compact subspace of $\mathbb{R}^N$ has topological dimension $N$.

**Proof.** (that it’s $\leq N$)

1. Break $\mathbb{R}^N$ into unit cubes. Let $J = \{(n, n + 1) | n \in \mathbb{Z}\}$ and $K = \{n | n \in \mathbb{Z}\}$. A M-cube is in the form $A_1 \times \cdots \times A_N$ where exactly $M$ of the sets are in $J$ and the rest are in $K$. Expand each cube into an open set so for given $M$, no two expanded $M$-cubes intersect. This is a open cover of order at most $m+1$.

2. Given an open covering $\{A_n\}$ of compact subspace $X$, shrink the cover above so unit cubes become $\frac{\delta}{N}$ cubes, where $\delta$ is a Lebesgue number of $X$ (a number such that every subset with diameter less than $\frac{\delta}{N}$ inside $X$ entirely inside some $A_n$), and intersect the sets in the two covers.

Points $\{v_0, \ldots, v_k\}$ in $\mathbb{R}^N$ are **affinely independent** if $\sum_{i=0}^k a_i v_i = 0, \sum_{i=1}^k a_i = 0$ imply each $a_i = 0$. A set $A$ of points is in general position in $\mathbb{R}^N$ if every subset containing at most $N + 1$ points is affinely independent.

**Imbedding Theorem:** Every compact metrizable space $X$ of topological dimension $m$ can be imbedded in $\mathbb{R}^{2m+1}$. (Ex. Graphs can be imbedded in $\mathbb{R}^3$.)

**Proof.**

1. Use the square metric on $\mathbb{R}^{2m+1}$: $|x - y| = \max\{|x_i - y_i|\}$. Use the sup metric on $C(X, \mathbb{R}^N)$: $\rho(f, g) = \sup\{|f(x) - g(x)|; x \in X\}$. Given continuous $f : X \to \mathbb{R}^{2m+1}$, define $\Delta(f) = \sup\{diam f^{-1}(\{z\})|z \in f(X)\}$ (how far $f$ deviates from being injective). Define $U_\varepsilon = \{f \in C(X, \mathbb{R}^{2m+1})|\Delta(f) < \varepsilon\}$.

2. $U_\varepsilon$ is open and dense. To show it’s open, given $f \in U_\varepsilon$, bound $|f(x) - f(y)|$ on $A = \{(x, y) | d(x, y) \geq b\}$. To show it’s dense, given $f \in C(X, \mathbb{R}^{2m+1})$, cover $X$ with finitely many open sets $\{U_i\}$ so that
   a. $diam(U_i) < \frac{\varepsilon}{2}$ in $X$
   b. $diam f(U_i) < \frac{\delta}{2}$ in $\mathbb{R}^{2m+1}$
   c. $\{U_i\}$ has order at most $m + 1$.

Take a partition of unity $\phi_i$ dominated by $\{U_i\}$. For each $i$ choose $x_i \in U_i$ and a point $z_i \in \mathbb{R}^{2m+1}$ within $\frac{\delta}{2}$ of the point $f(x_i)$ such that $\{z_1, \ldots, z_n\}$ is in general position in
\( \mathbb{R}^{2m+1} \) (possible by induction and fact that \( \mathbb{R}^{2m+1} \) is Baire space). Define \( g(x) = \sum_i \phi_i(x)z_i \). We have \( \rho(f, g) < \delta \) and \( g \in U_k \): If \( g(x) = g(y) \) then \( \sum_i [\phi_i(x) - \phi_i(y)]z_i = 0 \). Since \( \{U_i\} \) has order at most \( m + 1 \), at most \( 2(m + 1) \) of the coefficients are nonzero. Since \( z_i \) are in general position in \( \mathbb{R}^{2m+1} \), this forces all coefficients to be 0. Then \( y \in U_i \).

3. By Baire, \( \bigcap_{n=1}^{\infty} U_{1/n} \) is dense in \( C(X, \mathbb{R}^{2m+1}) \) and nonempty. Any function in this intersection gives an imbedding.

**Cor.** Every compact \( m \)-manifold has dimension equal to \( m \) so can be imbedded in \( \mathbb{R}^{2m+1} \). If \( X \) is compact metrizable, \( X \) can be imbedded in some \( \mathbb{R}^N \) iff \( X \) has finite topological dimension.