Geometry	Math Notes • Study Guide
	# Topology

1 Topological Spaces

1-1 Topologies

A **topology** on a set X is a collection $\mathcal{T}$ of subsets, called **open sets** satisfying:

1. $\phi, X \in \mathcal{T}$
2. The union of an arbitrary collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of a finite number of sets in $\mathcal{T}$ is in $\mathcal{T}$.

$(X, \mathcal{T})$ is called a topological space.

A is **closed** if X-A is open.

1. X, ϕ are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

If $\mathcal{T}$, $\mathcal{T}'$ are two topologies for X, and

1. $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}$ is coarser than $\mathcal{T}'$.
2. $\mathcal{T} \supseteq \mathcal{T}'$, then $\mathcal{T}$ is finer than $\mathcal{T}'$.
3. Either way, $\mathcal{T}, \mathcal{T}'$ are comparable.

Examples: (Collection of open sets)

1. Discrete topology: Collection of all subsets of X
2. Trivial topology: Collection only containing X, ϕ.
3. Finite complement topology: Collection of all subsets U with X-U finite, plus ϕ.

1-2 Bases

A **base** for a topology on X is a collection $\mathcal{B}$ of subsets, called base elements, of X such that any of the following equivalent conditions is satisfied.

1. Both the following are true.
   a. For each $x \in X$, there is at least one base element containing x.
   b. If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $x \in B_3 \subseteq B_1 \cap B_2$ for some $B_3 \in \mathcal{B}$.
2. For each open set U of X and each $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.
3. $\mathcal{T}$ (the open sets) is the collection of all unions of elements of $\mathcal{B}$. (Thus the base determines the topology.)

Let $\mathcal{B}, \mathcal{B}'$ be bases for $\mathcal{T}, \mathcal{T}'$ on X. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and each base element $B \in \mathcal{B}$ containing x, there is a base element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Examples:

1. The standard topology on the real line is generated by open intervals (a, b).
2. The lower limit topology $\mathbb{R}_l$ is generated by half-open intervals $[a, b)$.

A **subbase** $\mathcal{S}$ for a topology is a collection of subsets of X whose union equals X, and where $\mathcal{T}$ consists of all unions of finite intersections of elements in $\mathcal{S}$.

Let X be a totally ordered set, and define:

1. Open interval: $(a, b) = \{x | a < x < b\}$

	2. Half-open interval: $(a, b] = \{x \mid a < x \leq b\}, [a, b) = \{x \mid a \leq x < b\}$ 3. Closed interval: $[a, b] = \{x \mid a \leq x \leq b\}$ The order topology on X is the topology generated by the base containing… 1. Open intervals (a, b) 2. Intervals of the form $[a_0, b)$, $a_0 = \min X$, if it exists 3. Intervals of the form $(a, b_0]$, $b_0 = \max X$, if it exists
1-3	Limit Points and Convergence Let A be a subset of X. 1. The **interior** $\mathrm{Int}(A) = A^\circ$ of X is the union of all open sets contained in A. 2. The **closure** $\bar{A} = [A]$ of X is the intersection of all closed sets containing A. If A is open, $\mathrm{Int}(A) = A$ and if A is closed, $A^\circ = A$. Let Y be a subspace of X. Then the closure of A in Y is $\bar{A} \cap Y$. A subset $A \subseteq X$ is **dense** in X if $\bar{A} = X$. An open set containing x is a **neighborhood** of x. 1. $x \in \bar{A}$ iff any neighborhood of x intersects A. 2. Given a base for X, $x \in \bar{A}$ iff every base element B containing x intersects A. x is a **limit point** of A if every neighborhood of x intersects A in some point other than x. Let A' be the set of all limit points; then $\bar{A} = A \cup A'$. A is closed iff it contains all of its limit points. $\{x_n\}_{n=1}^{\infty}$ **converges** to x if for any neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$. Note that a sequence may converge to more than one point, since one-point sets may not be closed.
1-4	Product and Subspace Topology For topological spaces X, Y, the **product topology** on $X \times Y$ is the topology with base consisting of all the subsets of the form $U \times V$, where U, V are open subsets of X, Y, respectively. If $\mathcal{B}$ is a base for X and $\mathcal{C}$ a base for Y, then $$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$ is a base for the topology for $X \times Y$. $\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$ is a subbase for the product topology. Ex. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the standard topology for $\mathbb{R}^2$. If $Y \subseteq X$, the collection $$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$ is the **subspace topology**. If $A \subseteq X, B \subseteq Y$, then the product topology $A \times B$ is the subspace topology of $A \times B$ in $X \times Y$. A is closed in Y iff it is the intersection of a closed set in X with Y. If Y is closed in X, then $A \subseteq Y$ is closed in Y iff it is closed in X. Let X be a totally ordered set and Y be a subset. If Y is convex (i.e. $a, b \in Y$ imply $(a, b) \in Y$) then the order topology on Y is the same as the subspace topology.

1-5 Infinite Product Topology, Tychonoff Theorem

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of sets. Consider their Cartesian product $P = \prod_{\alpha \in J} X_\alpha$. (If all the sets are the same, this is denoted X^J.)

1. The **box topology** is generated by the base of sets $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α.
2. The **product topology** generated by the subbase
$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$
where $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) | U_\beta \text{ open in } X_\beta\}$ and π_β is the projection map $P \to X_\beta$. In other words, it is generated by the base of sets $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α and $U_\alpha \neq X_\alpha$ only for a finite number of α.

For finite products the two topologies are the same. The box topology is finer than the product topology.
In either topology, $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

<u>Tychonoff Theorem:</u> An arbitrary product of compact spaces $X = \prod_{\alpha \in J} X_\alpha$ is compact in the product topology.
<u>*Pf.*</u> Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property; we need to show $\bigcap \mathcal{A} \neq \phi$. By Zorn's Lemma there is a maximal collection $\mathcal{D}$ of subsets of X with the finite intersection property and which contains $\mathcal{A}$. By compactness of X_α, choose $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$; let $x = (x_\alpha)_{\alpha \in J}$. Any subbase element containing x intersects every element of $\mathcal{D}$. By maximality of $\mathcal{D}$, any finite intersection of elements of $\mathcal{D}$, and any subset that intersects every element of $\mathcal{D}$, is an element of $\mathcal{D}$. Then every subbase element $\pi_\beta^{-1}(U_\beta)$ containing x, and every base element containing x, belongs to $\mathcal{D}$. Then every base element containing x intersects every element of $\mathcal{D}$, i.e. any neighborhood of x intersects $\bigcap \mathcal{D}$. Hence $x \in \bigcap_{D \in \mathcal{D}} \overline{D} \subseteq \bigcap \mathcal{A}$.
(The proof for finite products does not need Axiom of Choice.)

1-6 Axioms of Separation

Axioms of separation:

1. For every pair $x, y \in X$ of distinct points, there exists a neighborhood of x not containing y. Equivalently, every finite point set is closed.
2. For every pair $x, y \in X$ of distinct points, there exist neighborhoods O_x, O_y of x and y that are disjoint.

A topological space X is a **T_1 space** if it satisfies the first axiom of separation and a **Hausdorff (T_2) space** if it satisfies the second axiom of separation.
In a **regular (T_3) space**, 1-point sets are closed and for $x \in X$ and closed $B \subseteq X$ not containing x, there exist disjoint open sets containing x and B.
In a **completely regular ($T_{3.5}$) space**, 1-point sets are closed and for each $x \in X$ and closed $B \subseteq X$ not containing x, there exists a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f(A) = \{0\}$.
In a **normal (T_4) space**, for every pair of disjoint closed sets $F_1, F_2 \subseteq X$ there exists disjoint open sets $O_1 \supseteq F_1$ and $O_2 \supseteq F_2$.
In a **completely normal (T_5) space**, every subspace is normal.

A space is completely regular iff it is the subspace of a normal space.

If X is a T_1 space, then x is a limit point of $A \subseteq X$ iff every neighborhood of x contains

	infinitely many points of A. In a Hausdorff space, every sequence of points in X converge to at most 1 point (called the limit). The following are Hausdorff: 1. Simply ordered sets in the order topology 2. Product of Hausdorff spaces (with either the box or product topologies) 3. Subspace of Hausdorff spaces A subspace of a regular space is regular and a product of regular spaces is regular.
1-7	Countability Axioms X has a **countable neighborhood base** at x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. **Countability axioms**: 1. X has a countable neighborhood base at each point. (First-countable) 2. X has a countable base for its topology. (Second-countable) X is a **Lindelöf space** if every open cover of X contains a countable subcover. X is **separable** if it has a countable dense subset. A subspace of a 1^{st} (2^{nd})-countable space is 1^{st} (2^{nd})-countable, and a countable product of 2^{nd}-countable space is 2^{nd}-countable. If X is 2^{nd}-countable (has a countable base) then it is Lindelöf and separable. All three conditions are equivalent if X is a metric space.
1-8	Continuous Functions Let X, Y be topological spaces. $f: X \to Y$ is **continuous** if for each open subset V of Y, $f^{-1}(V)$ is an open subset of X. It suffices to show that $f^{-1}(V)$ is open for every subbase element. The following are equivalent: 1. f is continuous. 2. For every subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$. 3. For every closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X. 4. For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$. (f is continuous at each point x) If f is bijective and f, f^{-1} are both continuous, then f is a **homeomorphism**. Here $U \subseteq X$ is open iff $f(U)$ is open. If $f: X \to Y$ is a homeomorphism when the range is restricted to $f(X)$, then f is an **imbedding** of X in Y. Constructing continuous functions 1. Constant functions $f(x) \equiv y_0$ are continuous. 2. If $A \subseteq X$ the inclusion function $i: A \to X$ is continuous. 3. If $f: X \to Y$ and $g: Y \to Z$ are continuous, then so is $g \circ f$. 4. The restriction $f\|_A$ of a continuous function f to a subspace $A \subseteq X$ is continuous. 5. If $f: X \to Y$ is continuous, and $f(X) \subseteq Z$, then $f: X \to Z$ is continuous. (change range) 6. (Local formulation of continuity) $f: X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f\|_{U_\alpha}$ is continuous for each α.

	7. (Pasting lemma) Let $X = A \cup B$ where A, B are closed in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$ then f and g combine to give a continuous function $h: X \to Y$: $$h(x) = \begin{cases} f(x), x \in A \\ g(x), x \in B \end{cases}$$ 8. Let $f: A \to \prod_{\alpha \in J} X_\alpha$ be given by $f(a) = \left(f_\alpha(a)\right)_{\alpha \in J}$. Then f is continuous iff $f_\alpha: A \to X_\alpha$ is continuous for all α.
1-10	Quotient Topology A surjective map $p: X \to Y$ is a **quotient map** when $U \subseteq Y$ is open in Y iff $p^{-1}(U)$ is open in X. (Equivalently, $A \subseteq Y$ is closed in Y iff $p^{-1}(A)$ is closed in X.) An **open map** sends open sets to open sets, and a **closed map** sends closed sets to closed sets. If A is a set and $p: X \to A$ is surjective, then the unique topology on A such that p is a quotient map is the **quotient topology** induced by p. It consists of the subsets $U \subseteq A$ such that $p^{-1}(U)$ is open in X. Let X^* be a partition of X into disjoint subsets whose union is X, and let $p: X \to X^*$ be the map carrying x to the element of X^* containing x. Then X^* with the quotient topology is called the **quotient space** of X. (X^* is obtained by identifying equivalent points.) A subset C of X is **saturated** with respect to p if C contains every set $p^{-1}(\{y\})$ that it intersects. If $A \subseteq X$ is saturated, then the restriction $p\|_A: A \to p(A)$ is a quotient map if A is open or closed, or p is an open or closed map. The composition of two quotient maps is a quotient map. If each class in X^* is closed, then X^* is a T_1 space. Let $p: X \to Y$ be a quotient map. Let $g: X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$. Then g factors through Y, i.e. there exists a map $f: Y \to Z$ so that $f \circ p = g$. f is continuous iff g is continuous; f is a quotient map iff g is a quotient map. If g is surjective and $Y = \{g^{-1}(\{z\}) \mid z \in Z\}$ is given the quotient topology, then f is bijective and continuous; f is a homeomorphism iff g is a quotient map. If Z is Hausdorff, then so is X^*.
1-11	Topological Groups A **topological group** G is a group that is also a T_1 space, such that the maps • $G \times G \to G: (x, y) \to xy$ • $G \to G: x \to x^{-1}$ are continuous. If H is a subgroup of G then collection of cosets G/H can be given the quotient topology.

Classification of Topological Spaces

Topological space X

Lindelöf:
Every open cover contains a countable subcover.

Separable: (H)
Contains a countable dense subset

1st axiom of countability: (H)
Every point x has a countable neighborhood base.

T_1 space (1st axiom of separation): (H)
For every pair $x, y \in X$ of distinct points, there exists a neighborhood of x not containing y.

2nd axiom of countability: (H)
A countable base exists.

Manifold: (metrizable)
Hausdorff, 2nd-countable space such that each point has a neighborhood homeomorphic with an open subset of $\mathbb{R}^n$

T_2 (Hausdorff) space (2st axiom of separation): (H)
For every pair $x, y \in X$ of distinct points, there exist disjoint neighborhoods O_x, O_y of x and y.

Sequentially compact:
every sequence of points has a convergent subsequence

Countably (limit point) compact: (3 equivalent definitions)

- Every infinite subset of X has a finite subcover at least 1 limit point in X.
- Every countable open cover of X has a finite subcover.
- Every countable centered system of closed subsets has a non-empty intersection.

T_3 (Regular) space: (H)
1-point sets are closed and for $x \in X$ and closed $B \subseteq X$ not containing x, there exist disjoint open sets containing x and B.

Compactly generated:
If $A \cap C$ is open in C for each compact C, then A is open in X.

Locally compact:
For each x there is some compact subspace C of X that contains a neighborhood of x

$T_{3.5}$ (Completely regular, Tychonoff) space: (H)
1-point sets are closed and for each $x \in X$ and closed $B \subseteq X$ not containing x, there exists a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f(A) = \{0\}$.

Compact: (2 equivalent definitions)
(H-closed subset, I)

- Every open cover of X has a finite subcover.
- Every countable centered system of closed subsets has a non-empty intersection.

Paracompact: (H-closed subset)
Every open covering of X has a locally finite open refinement that covers X.

T_4 (Normal) space: (H-closed subset)
For every pair of disjoint closed sets $F_1, F_2 \subseteq X$ there exists disjoint open sets $O_1 \supseteq F_1$ and $O_2 \supseteq F_2$.

Compactum: (H-closed subset, I)
Compact Hausdorff space

Topological group:
Has group structure; multiplication and inverses are continuous.

T_5 (Completely normal) space: (H)
Every subspace is normal.

Metric space (metrizable)

Baire space:
Given any countable collection $\{A_n\}$ of closed sets of X with empty interior, their union $\cup_n A_n$ has empty interior.

Locally metrizable:
Every point has a neighborhood that is metrizable in the subspace topology.

Totally bounded:
Has a finite ε-net (can be covered with a finite number of ε-balls for any $\varepsilon > 0$)

Complete:
Every Cauchy sequence converges to an element in X.

Locally (path) connected: (NH)
For every neighborhood U of an element x there is a (path) connected neighborhood V of x contained in U

Path connected: (NH)
every pair of points in X can be joined by a path in X

Connected: (NH)
No subsets other than ϕ and X are both open and closed.

- Regular + 2nd countable ⇒ normal
- Urysohn metrization: Normal + 2nd countable ⇒ metrizable
- Nagata-Smirnov metrization: regular + has countably local base = metrizable
- Smirnov metrization: Locally metrizable + paracompact = metrizable
- In a metric space, compact = countably compact = sequentially compact.
- Paracompact + Hausdorff ⇒ normal
- Regular + Lindelöf ⇒ paracompact

2 Compactness

2-1 Compactness

An **open cover** of a set E in a topology X is a collection $\mathcal{F}$ of open subsets such that $E \subseteq \bigcup_{G \in \mathcal{F}} G$. A subset $K \subseteq X$ is **compact** if every open cover of K contains a finite subcover. K is **sequentially (or countably) compact** if every infinite subset of K has a limit point in K.

On subsets:

- Suppose $K \subseteq Y \subseteq X$. Then K is compact relative to X if it is compact relative to Y. In other words, compactness is an intrinsic property.

1. Closed subsets of compact sets are compact.
2. Compact subsets of Hausdorff spaces are closed. (Pf. If Y is a compact subspace of the Hausdorff space X and $x_0 \notin Y$ there exist disjoint open sets U and V containing x_0 and Y, respectively.)

The image of a compact space under a continuous map is compact. (Pf. For an open cover of $f(X)$, take the inverse of each subset. They're open since f is continuous; choose a finite subcover and take the image.)
If $f: X \to Y$ is a bijective continuous function, X is compact, and Y is Hausdorff, then f is a homeomorphism.

A collection $\mathcal{C}$ of subsets of X has the **finite intersection property** (or is **centered**) if for every finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$, $\bigcap_{i=1}^{n} C_i$ is nonempty. (Pf. Prove the contrapositive with the complements.)
X is compact iff for every collection $\mathcal{C}$ of closed sets with the finite intersection property, $\bigcap_{C \in \mathcal{C}} C$ is nonempty.
In particular, any nested sequence of closed nonempty sets in a compact set X has a nonempty intersection. For $\mathbb{R}$ this gives the Nested Intervals Theorem.

In a simply ordered set with the least upper bound property, each closed interval is compact. In particular, closed intervals are compact in $\mathbb{R}$. (Pf. Consider the set of points $y > a$ such that $[a, y]$ can be covered by finitely many elements of the open cover A.)
<u>Extreme Value Theorem:</u> If X is compact, f is compact, and Y has the order topology, $f: X \to Y$ attains its maximum and minimum.

A point x of X is an **isolated point** if $\{x\}$ is open in X.
A nonempty compact Hausdorff space with no isolated points is uncountable. Therefore any closed interval in $\mathbb{R}$ is uncountable.
Pf. Given a countable sequence of points $\{x_n\}$, build a nested sequence of open sets $V_1 \supseteq V_2 \supseteq \cdots$ such that $x_n \notin \overline{V_n}$. $x \in \bigcap \overline{V_n}$ (nonempty by compactness) cannot be in $\{x_n\}$.

2-2 Limit Point Compactness

X is **limit point compact** if every infinite subset of X has a limit point. X is **sequentially compact** if every sequence of points of X has a convergent subsequence.
Compactness implies limit point compactness.

All three notions of compactness are equivalent for a metric space (see analysis notes).

2-3 Local Compactness and Compactification

X is **locally compact** at x if there is some compact subspace C of X that contains a neighborhood of x. X is said to be locally compact if it is locally compact at every point.

A **compactification** of a space X is a compact Hausdorff space Y containing X such that $\bar{X} = Y$. Two compactifications Y_1, Y_2 are equivalent if there is a homeomophism $h: Y_1 \to Y_2$ such that $h(x) = x$ for every $x \in X$.

X is locally compact Hausdorff iff it has a **one-point compactification**, i.e. a compactification Y such that $Y - X$ consists of a single point. Y is unique up to equivalence.
Pf.

- Uniqueness: Let $h: Y \to Y'$ be the map that is the identity on X and sends $p \in Y - X$ to $q \in Y' - X$. Check that it maps an open set to an open set; if U contains p note Y-U is closed and hence compact; its image is compact and hence closed.
- Existence: Adjoin an object ∞ to X. Let the open sets be
    1. Open sets in X, and
    2. All sets of the form $Y - C$, C compact subspace of X.C

  Check that X does have the subspace topology. If $\mathcal{A}$ is an open cover of Y, then $\mathcal{A}$ must contain a set of type 2. Finitely many of the other subsets cover C; add C. Y is compact: just note if $y = \infty$ then take a compact set C containing a neighborhood U of x; U and Y-C are disjoint neighborhoods.
- Converse: Choose disjoint U, V containing x, p. $Y - V$ is compact and contains U in X.

Let X be Hausdorff. X is locally compact iff given $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq U$.
An open or closed subset of a locally compact Hausdorff set is locally compact.
A space is homeomorphic to an open subspace of a compact Hausdorff space iff it is locally compact Hausdorff.

2-4 Stone-Čech Compactification

Let X be completely regular. There exists an unique compactification (called the **Stone-Čech compactification** $\beta(X)$) Y of X such that every bounded continuous map $f: X \to \mathbb{R}$ extends uniquely to a continuous map of Y into $\mathbb{R}$.
Pf.

1. Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded continuous real-valued functions on X; let $I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)]$. Define $h: X \to \prod_{\alpha \in J} I_\alpha$ by $h(x) = \left(f_\alpha(x)\right)_{\alpha \in J}$. By Tychonoff, $\prod_{\alpha \in J} I_\alpha$ is compact. h is an imbedding by the Imbedding Theorem.
2. The compactification Y of X can be identified with $\overline{h(X)}$. Let $H: Y \to \prod_{\alpha \in J} I_\alpha$ be its imbedding into $\prod_{\alpha \in J} I_\alpha$.
3. Any bounded continuous function f_α can be extended to $\pi_\alpha \circ H$.
4. (Uniqueness of extension) Let $A \subseteq X$, let Z be Hausdorff, and let $f: A \to Z$ be continuous. There is at most one extension of f to a continuous function $\bar{A} \to Z$. Pf. Suppse there were two g, g' with $g(x) \neq g'(x)$. Take disjoint neighborhoods U, U' containing $g(x), g'(x)$ and by continuity take O_x so $g(O_x) \subseteq U, g'(O_x) \subseteq U'$. But some point in A is in O_x, contradiction.
5. (Uniqueness of compactification)
    a. Given any continuous map $f: X \to C$ into compact Hausdorff space C, f

	extends uniquely to a continuous map $g: \beta(X) \to C$. (*Pf.* Since C is completely regular it can be imbedded in $[0,1]^J$.) b. Suppose Y_1, Y_2 are compactifications. Using (a) on $i_2: X \to Y_2$, there is a continuous map $f_2: Y_1 \to Y_2$, and vice versa. The composition is the identity on X; by uniqueness of extension it must be the identity.
2-5	Paracompactness X is **paracompact** if every open cover of X has a locally finite open refinement that covers X. Every paracompact Hausdorff space is normal. (first show it's regular) Every closed subspace of a paracompact space is paracompact. Let X be regular. The following conditions are equivalent: Every open cover of X has a refinement that is… 1. An countably locally finite open cover of X. 2. A locally finite cover of X. 3. A locally finite closed cover of X. 4. A locally finite open cover of X. (1)⇒(2): Write $\mathcal{B} = \bigcup_n \mathcal{B}_n$, where $\mathcal{B}_n$ is locally finite. Define $S_n(U) = U - \bigcup_{i<n, U' \in \mathcal{B}_i} U'$ (to make sets in different $\mathcal{C}_n$ disjoint—shrinking trick). Let $\mathcal{C}_n = \{S_n(U) \mid U \in \mathcal{B}_n\}$; take $\mathcal{C} = \bigcup_n \mathcal{C}_n$. (2)⇒(3): Let $\mathcal{A}$ be an open cover. Let $\mathcal{B} = \{U \text{ open} \mid \overline{U} \in A \text{ for some } A \in \mathcal{A}\}$. By regularity, $\mathcal{B}$ covers X. Refine $\mathcal{B}$ to locally finite $\mathcal{C}$ by (2); take $\mathcal{D} = \{\bar{C} \mid C \in \mathcal{C}\}$. (3)⇒(4): Let $\mathcal{A}$ be open cover. Take locally finite refinement $\mathcal{B}$ that covers X. Introduce an auxiliary locally finite closed cover $\mathcal{C}$ and use it to expand elements of $\mathcal{B}$ into open sets: By local finiteness of $\mathcal{B}$, the collection of all open sets that intersect finitely many elements of $\mathcal{B}$ is an open cover. Use (3) to get a closed locally finite refinement $\mathcal{C}$ covering X. Let $\mathcal{C}(B) = \{C \mid C \in \mathcal{C} \text{ and } C \subseteq X - B\}$ and $E(B) = X - \bigcup_{C \in \mathcal{C}(B)} C$. To make $\{E(B)\}$ a refinement, for each $B \in \mathcal{B}$ choose $F(B) \in \mathcal{A}$ containing B. Let $\mathcal{D} = \{E(B) \cap F(B) \mid B \in \mathcal{B}\}$. Every metrizable space is paracompact. (Pf. Get a countably locally finite refinement cover and use above.) Every regular Lindelöf space is paracompact.

3 Metric Topologies and Continuous Functions

3-1 Metric Spaces

A set X with a real-valued function (a metric) $d(p,q)$ on pairs of points in X is a **metric space** if:

1. $d(p,q) \geq 0$ with equality iff $p = q$.
2. $d(p,q) = d(q,p)$
3. $d(p,q) \leq d(p,r) + d(r,q)$ (Triangle inequality)

A **metric space** is a topology with base consisting of ε-neighborhoods

$$N(p,\varepsilon) = \{p \in X | d(p,q) < \varepsilon\}.$$

X is **metrizable** if there exists a metric d on X that induces the topology of X. X is **topologically complete** if there exists a metric so that it is a complete metric space.

Let d, d' be metrics on X inducing topologies $\mathcal{T}, \mathcal{T}'$. Then $\mathcal{T}$ is finer than $\mathcal{T}'$ iff for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that $N_{d'}(x,\delta) \subseteq N_d(x,\varepsilon)$.

3-2 Metrics for $\mathbb{R}^J$

The metric space $\mathbb{R}^n$, with the Euclidean metric

$$d(x,y) = \left(\sum_{i=1}^{n} |x_k - y_k|^2\right)^{\frac{1}{2}}$$

or with the square metric

$$d(x,y) = \max|x_k - y_k|$$

induce the product topology.

Let $\bar{d}(x,y) = \min(|x-y|, 1)$, and define the **uniform metric** $\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) | a \in J\}$ on $\mathbb{R}^J$. The uniform topology is finer than the product topology but coarser than the box topology. (They are all different if J is infinite.)

If J is countable then the metric

$$D(x,y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

induces the product topology on $\mathbb{R}^\omega$. When J is infinite, $\mathbb{R}^J$ is not metrizable when J is uncountable or when the box topology is used.

3-3 Continuous Functions in Metric Spaces

The topological and analytic definitions of continuity are equivalent: let X, Y be metric spaces; continuity of $f: X \to Y$ is equivalent to the requirement that for any $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x,y) < \delta$ implies $d_Y\big(f(x), f(y)\big) < \varepsilon$.

<u>Sequence lemma:</u> Let X be a topological space and $A \subseteq X$. If there is a sequence of points of A converging to x then $x \in \bar{A}$; the converse holds if X has a countable basis at x (in particular, it holds when X is metrizable).

Let $f: X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X the sequence $f(x_n)$ converges to $f(x)$. The converse holds holds if X has a countable basis at x.

	Uniform limit theorem: Let X be a topological space, Y be a metric space, and $f_n: X \to Y$ be a sequence of continuous functions. If (f_n) converges uniformly to f, then f is continuous.
3-4	Urysohn Metrization Theorem Urysohn Lemma: Let X be a normal space and A, B be disjoint closed subsets of X. For any $a \le b$ there exists a continuous map $f: X \to [a,b]$ such that $f(x) = a$ for every $x \in A$ and $f(x) = b$ for every $x \in B$. (Any two closed sets can be separated by a continuous function.) *Pf.* Order the rational numbers in [0,1]: $\{a_n\}_{n\ge 0} = 1, 0, \ldots$ By normality, given any set S and open T such that $\bar{S} \subset T$ there exists an open set U such that $\bar{S} \subset U, \bar{U} \subset T$. Let $U_1 = X - B$, $A \subset U_0$ and inductively define open sets U_{a_i} such that $U_p \subset U_q$ when $p < q$. Set $U_p = \phi$ for $p < 0$ and $U_p = X$ for $p > 1$. Let $f(x) = \inf\{p \mid x \in U_p\}$. This gives $f: X \to [0,1]$. Imbedding Theorem: Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $X \to \mathbb{R}$ such that for each point x_0 and each neighborhood of x_0 there is an index α such that $f_\alpha(x_0) > 0$ and $f(X - U) = \{0\}$. Then the function $F: X \to \mathbb{R}^J$ defined by $$F(x) = \left(f_\alpha(x)\right)_{\alpha \in J}$$ is an imbedding of X in $\mathbb{R}^J$. *Pf.* Need to show the inverse of F is continuous; i.e. if U is open then $F(U)$ is open. Let $z_0 \in F(U)$. Choose N so that $f_N(x_0) > 0, f_N(X - U) = \{0\}$. Let $W = \pi_N^{-1}\left((0,\infty)\right) \cap F(X)$; then $z_0 \in W \subseteq F(U)$. Urysohn Metrization Theorem: Every regular space X with a countable base (i.e. X is 2nd countable) is metrizable. *Pf.* There exists a countable collection of continuous functions as in the Imbedding Theorem: Take a countable base $\{B_n\}$; for every pair (m,n) so that $\overline{B_n} \subseteq B_m$, use Urysohn to get continuous $g_{n,m}$ with $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X - B_m) = \{0\}$. Use the Imbedding Theorem.
3-5	Tietze Extension Theorem Tietze Extension Theorem: Let X be normal and let A be a closed subspace of X. Any continuous map $f: A \to Y$ where $Y = [a,b]$ or $\mathbb{R}$ may be extended to a continuous map of all X into Y. *Pf.* First prove it for $[-1,1]$. Let $I_1 = \left[-r, -\frac{1}{3}r\right], I_2 = \left[-\frac{1}{3}r, \frac{1}{3}r\right], I_3 = \left[\frac{1}{3}r, 1\right]$. By Urysohn Lemma, there is a continuous function $g: X \to \left[-\frac{1}{3}r, \frac{1}{3}r\right]$ with $g(x) = -\frac{1}{3}r$ for $x \in f^{-1}(I_1)$ and $g(x) = \frac{1}{3}r$ for $x \in f^{-1}(I_3)$. (g is not too large but approximates f.) Apply this result for $r = 1$ and f to get g_1, then inductively for $r = \left(\frac{2}{3}\right)^n$ and $f - g_1 - \cdots - g_n$ to get g_{n+1} (Approximate the error in the previous approximation). Then $\sum_{n=1}^{\infty} g_n(x)$ is the desired function; it is equal to f on A and continuous by the Weierstrass M-test. For $\mathbb{R}$, consider the homeomorphic space $(-1,1)$. Define g as above; let $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$. By Urysohn, there is continuous $\phi: X \to [0,1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. The desired function is $\phi(x)g(x)$. (it's equal to g on A, and doesn't take the value 1.)

3-6

Nagata-Smirnov Metrization

A collection $\mathcal{A}$ of subsets of X is **locally finite** in X if every point of X has a neighborhood intersecting only finitely many elements of $\mathcal{A}$.
If $\mathcal{A}$ is locally finite, then $\{\bar{A} | A \in \mathcal{A}\}$ is locally finite, and $\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \bar{A}$.

$\mathcal{B}$ is **countably locally finite** if $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.

Let $\mathcal{A}$ be a collection of subsets of X. A collection $\mathcal{B}$ **refines** $\mathcal{A}$ if for each element $B \in \mathcal{B}$, there is an element A of $\mathcal{A}$ containing $\mathcal{B}$. $\mathcal{B}$ is open/ closed if all sets in $\mathcal{B}$ are open/ closed.

Let X be a metric space. If $\mathcal{A}$ is an open cover of X, then there is an open cover $\mathcal{B}$ of X refining $\mathcal{A}$ that is countably locally finite.
Pf. Well-order $\mathcal{A}$. For $U \subseteq X$ define $S_n(U) = \{x | N(x, 1/n) \subseteq U\}$ (shrinking by $1/n$). Define

$$T_n(U) = S_n(U) - \bigcup_{V<U} V.$$

(Exclude other sets for local finiteness.) Let $E_n(U) = \bigcup_{x\in T_n(U)} N\left(x, \frac{1}{3n}\right)$ (expand $T_n(U)$ by $\frac{1}{3n}$; we have $E_n(U), E_n(V)$ disjoint when $U \neq V$; note $E_n(U)$ is open). Let $\mathcal{B}_n = \{E_n(U) | U \in \mathcal{A}\}$.

A subset $A \subseteq X$ is a **G_δ-set** if it is the intersection of a countable collection of open subsets of X.

Let X be a regular space with countably locally finite base. Then X is normal, and every closed set in X is a G_δ-set in X.
Pf.

1. Let W be open. We show there is a countable collection of open sets U_n so $W = \bigcup U_n = \bigcup \overline{U_n}$. Write $B = \bigcup_n \mathcal{B}_n$ as a union of locally finite collections. Let $\mathcal{C}_n = \{B \in \mathcal{B}_n | \bar{B} \subseteq W\}$. Let $U_n = \bigcup_{B\in\mathcal{C}_n} B$.
2. Given closed C, write $X - C = \bigcup \overline{U_n}$ by (1). Then $C = \bigcap_n X - \overline{U_n}$.
3. X is normal: Let C, D be disjoint closed sets. By (1) write $X - D = \bigcup U_n = \bigcup \overline{U_n}$ and $X - C = \bigcup V_n = \bigcup \overline{V_n}$; then $U = \bigcup_{n=1}^{\infty}(U_n - \bigcup_{i=1}^{n} \overline{V_i})$ and $V = \bigcup_{n=1}^{\infty}(V_n - \bigcup_{i=1}^{n} \overline{U_i})$ are disjoint open sets around C, D.

Let X be normal and A a closed G_δ-set in X. Then there is a continuous function $f: [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$.
Pf. Write $A = \bigcup U_n$.By Urysohn Lemma choose $f_n: X \to [0,1]$ so $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \in X - U_n$. Take $f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}$.

<u>Nagata-Smirnov Metrization Theorem:</u> X is metrizable iff X is regular and has a countably locally finite base $\mathcal{B}$.
Pf.

1. X is normal and every closed set in X is a G_δ-set.
2. Write $\mathcal{B} = \bigcup \mathcal{B}_n$. Let J be the set of pairs $(n, B \in \mathcal{B}_n)$. For each pair choose continuous $f_{n,B}: X \to [0,1/n]$ so $f_{n,B}(x) > 0$ for $x \in B$ and $f_{n,B}(x) = 0$ for $x \notin B$. Then $\{f_{n,B}\}$ separates points from closed sets in X.
3. Define $F: X \to [0,1]^J$ by $F(x) = \left(f_{n,B}(x)\right)_{(n,B)\in J}$. F is an imbedding in the product topology.

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4. To show F is an imbedding in the uniform topology, we need to show F is continuous. Note on $[0,1]^J$, $\rho\big((x_\alpha),(y_\alpha)\big) = \sup\{|x_\alpha - y_\alpha|\}$. Take $x_0 \in X$. By local finiteness, choose a neighborhood U_n of x_0 intersecting finitely many sets in $\mathcal{B}_n$. Then only finitely many of the $f_{n,B}$ are nonzero. By continuity, choose neighborhood V_n of x so that the nonzero functions vary by less than ε. Choose N so $\frac{1}{N} < \varepsilon$. Let $W = \bigcap_{i=1}^{N} V_i$. Then for $x \in W$, $\left|f_{n,B}(x) - f_{n,B}(x_0)\right| < \varepsilon$ (for $n < N$, this is from $x \in V_n$, for $n \geq N$, this is from $f_{n,B}(x) < \frac{1}{N}$).
5. (Converse) Let $\mathcal{A}_m = \{N(x, 1/m) | x \in X\}$. There is an open covering $\mathcal{B}_m$ of X refining $\mathcal{A}_m$ that is countably locally finite. Let $\mathcal{B} = \bigcup_m \mathcal{B}_m$.

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Smirnov Metrization

<u>Smirnov Metrization Theorem:</u> X is metrizable iff it is a locally metrizable paracompact Hausdorff space.
<u>*Pf.*</u> Cover X by metrizable open sets; choose a locally finite open refinement $\mathcal{C}$ that covers X. Take metrics $d_C: C \times C \to \mathbb{R}$. Let $\mathcal{A}_m = \{N_C(x, 1/m) | x \in C \text{ and } C \in \mathcal{C}\}$. By paracompactness let $\mathcal{D}_m$ be a locally finite open refinement covering X; then $\mathcal{D} = \bigcup_m \mathcal{D}_m$ is a countably locally finite base. By Nagata-Smirnov, X is metrizable.

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Topologies on Function Spaces, Arzela-Ascoli Theorem

X is **compactly generated** if whenever $A \cap C$ is open for every compact subspace C, we have that A is open in X. If X is locally compact, or X is 1st-countable, then X is compactly generated.
If X is compactly generated then $f: X \to Y$ is continuous iff for each compact subspace C of X, $f|_C$ is continuous.

On Y^X (the set of functions from X to Y), the **topology of pointwise convergence** is the topology generated by the subbase $S(x, U) = \{f | f(x) \in U\}$ where U is an open set in Y.

- A sequence of functions converges to f in the topology of pointwise convergence iff the sequence converges pointwise (i.e. $f_n(x) \to f(x)$ for each x)).

For a metric space Y, the **topology of compact** (or **uniform**) **convergence** on Y^X is the topology generated by the base $B_C(f, \varepsilon) = \left\{g \middle| \sup\left\{d\big(f(x), g(x)\big) \middle| x \in C\right\} < \varepsilon\right\}$.

- A sequence $f_n: X \to Y$ converges to f in this topology iff for each compact subspace $C \subseteq X$, the sequence $f_n|_C$ converges uniformly to $f|_C$.
- The space of continuous functions $\mathcal{C}(X, Y)$ is closed in the topology of compact convergence. (Pf. uniform limit theorem.)

A generalization of the topology of compact convergence to arbitrary X, Y is the **compact-open topology** on $\mathcal{C}(X, Y)$, generated by the subbase $S(C, U) = \{f | f(C) \subseteq U\}$.

- If Y is a metric space, then the compact-open topology and the topology of compact convergence coincide.

Let Y be a metric space. For the function space Y^X, the inclusions of topologies is
(pointwise convergence)⊆(compact convergence)⊆(uniform)
If X is compact, the right two coincide; if X is discrete, the first two coincide.

Note the compact convergence topology does not depend on the metric of Y.
Let X be locally compact Hausdorff, and let $\mathcal{C}(X, Y)$ have the compact-open topology. Then the evaluation map $e: X \times \mathcal{C}(X, Y) \to Y$ defined by $e(x, f) = f(x)$ is continuous.

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Let Y be a metric space, and let $\mathcal{F} \subseteq \mathcal{C}(X,Y)$. $\mathcal{F}$ is **equicontinuous** at x_0 if given $\varepsilon > 0$ there is a neighborhood U of x_0 such that for all $x \in U$ and all $f \in \mathcal{F}$, $d\big(f(x), f(x_0)\big) < \varepsilon$. If $\mathcal{F}$ is equicontinuous at every point of X, it is said to be equicontinuous.[1]

Arzela-Ascoli Theorem: Let Y be a metric space. Give $\mathcal{C}(X,Y)$ the topology of compact convergence and let $\mathcal{F}$ be a subset of $\mathcal{C}(X,Y)$.

1. If $\mathcal{F}$ is equicontinuous and the set $\mathcal{F}_a = \{f(a) | f \in \mathcal{F}\}$ has compact closure for each $a \in X$, then $\mathcal{F}$ is contained in a compact subspace of $\mathcal{C}(X,Y)$.
2. The converse holds if X is locally compact Hausdorff.

Pf.

1. $\mathcal{G} = \bar{\mathcal{F}}$ is a compact subspace of (Hausdorff) Y^X under the product (pointwise convergence) topology. Indeed, $\mathcal{G}$ is closed and contained in $\prod_{a \in X} \overline{\mathcal{F}_a}$, compact by Tychonoff.
2. $\mathcal{G}$ is equicontinuous. (ε argument)
3. The product topology on Y^X and the compact convergence topology on $\mathcal{C}(X,Y)$ coincide on $\mathcal{G}$. Suffices to show product topology is finer than the compact convergence topology. Given $B_C(g,\varepsilon)$, cover C by finitely many open sets U_i so that $d\big(g(x), g(x_i)\big) < \frac{\varepsilon}{3}$ for each $x \in U_i, g \in \mathcal{G}$. Let $B = \{h \in Y^X | \forall i, d\big(h(x_i), g(x_i)\big) < \frac{\varepsilon}{3}\}$ (a base element for Y^X); then $B \cap \mathcal{G} \subseteq B_C(g,\varepsilon) \cap \mathcal{G}$.
4. (part 2) Let $\mathcal{H}$ be a compact subspace of $\mathcal{C}(X,Y)$ containing $\mathcal{F}$. It suffices to show $\mathcal{H}$ is equicontinuous and $\mathcal{H}_a$ is compact for each $a \in X$. $\mathcal{H}_a$ is compact because it is the image of a continuous map from $\mathcal{H}$ (via the evaluation map). The restriction map $r: \mathcal{C}(X,Y) \to \mathcal{C}(A,Y)$ is continuous; thus $\mathcal{R} = \{f|_A : f \in \mathcal{H}\}$ is compact. The compact convergence and uniform topologies on $\mathcal{C}(A,Y)$ coincide.
    a. A metric space is compact iff it is complete and totally bounded.

Let X be a space and (Y,d) be a metric space. If $\mathcal{F} \subseteq \mathcal{C}(X,Y)$ is totally bounded under the uniform metric corresponding to d then $\mathcal{F}$ is equicontinuous under d.

By (a), $\mathcal{R}$ is totally bounded in the uniform metric on $\mathcal{C}(A,Y)$; by (b), $\mathcal{R}$ is equicontinuous relative to d.

[1] Warning: This is different from the analysis definition.

4	Connectedness
4-1	Connectedness A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. X is **connected** if it satisfies any of the two equivalent conditions: 1. There is no separation of X. 2. The only subsets of X that are both empty and closed are ϕ and X. If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. Y is connected if there is no separation of Y (under this definition). Basic results: 1. If C and D form a separation of X, and $Y \subseteq X$ is connected, then Y lies entirely within C or D. 2. The union of a collection of connected subspaces of X with a point in common is connected. 3. If A is connected and $A \subseteq B \subseteq \bar{A}$ then B is connected. 4. The image of a connected space under a continuous map is connected. 5. A finite Cartesian product of connected spaces is connected. Define $x{\sim}y$ if there is a connected subspace of X containing both x and y. The equivalence classes are the **connected components** of X. Equivalently, for $x \in E$, the connected component of x is the union of all connected subsets containing x. The connected components form a partition of E, they are all closed sets, and every connected subset of X is entirely within one of them. A simply ordered set L having more than one element is a **linear continuum** if 1. L has the least upper bound property. 2. If $x < y$ there exists z such that $x < z < y$. If L is a linear continuum in the order topology, then L is connected, as are intervals and rays in L. In particular, $\mathbb{R}$ and its intervals and rays are connected. <u>Intermediate Value Theorem:</u> Let $f: X \to Y$ be a continuous map where X is connected and Y has the order topology. If $a, b \in X$ and c is between $f(a), f(b)$, then there exists a point $c \in X$ with $f(c) = r$.
4-2	Path connectedness A **path** in X from x to y is a continuous map $f: [a, b] \to X$ such that $f(a) = x, f(b) = y$. X is **path connected** if every pair of points in X can be joined by a path in X. Any path connected space is connected, but not vice versa. *Ex.* Topologist's sine curve $\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \middle\| 0 < x \leq 1\right\}$ is connected but not path connected. Defining $x{\sim}y$ if there is a path from x to y, the equivalence classes are the **path components** of X. Each nonempty path connected subspace is entirely within one path component.

4-3	Local connectedness X is **locally (path) connected** at x if for every neighborhood U of x there is a (path) connected neighborhood V of x contained in U. If X is locally (path) connected at every point, it is simply said to be locally (path) connected. X is locally (path) connected iff for every open set U of X, each (path) component of U is open in X. Each path component of X lies in a component of X. If X is locally path connected, then the components and path components coincide.

5 Manifolds and Dimension

5-1 Baire Spaces

X is a **Baire space** if any of the two following conditions hold:

1. Given any countable collection $\{A_n\}$ of closed sets of X with empty interior, their union $\bigcup_n A_n$ has empty interior.
2. Given any countable collection $\{U_n\}$ of dense open sets of X, their intersection $\bigcap_n U_n$ is dense in X.

A subset of a space X is of the **first category** if it is contained in the union of a countable collection of closed sets having empty interior, and of the **second category** otherwise. In a Baire space, every nonempty open set is of the second category.

<u>Baire Category Theorem:</u> If X is a compact Hausdorff space or a complete metric space (such as $\mathbb{R}$) then X is a Baire space.
<u>Pf.</u> Given a countable collection $\{A_n\}$ of closed sets and a nonempty open set $U_0 \subseteq X$ we find $x \in U_0$ not in any of the A_n. Inductively define U_n: choose a point of U_{n-1} not in A_n, then choose U_n to be a neighborhood of this point so that $\overline{U_n} \cap A_n = \phi$, $\overline{U_n} \subseteq U_{n-1}$, and in the metric case, $\operatorname{diam}(U_n) < \frac{1}{n}$. If compactness is assumed, then $\bigcap_n \overline{U_n}$ is nonempty. In the metric case, get a Cauchy sequence.

Any open subspace of a Baire space is a Baire space.
Let Y be a metric space. Let $f_n: X \to Y$ be a sequence of continuous functions converging pointwise to f. If X is a Baire space the set of points at which f is continuous is dense in X.

5-2 Imbeddings of Manifolds

A **m-manifold** is a Hausdorff space with countable basis such that each point $x \in X$ has a neighborhood homeomorphic with an open subset of $\mathbb{R}^m$. A 1-manifold is a curve and a 2-manifold is a surface.

For $\phi: X \to \mathbb{R}$, the **support** of ϕ is defined as $\operatorname{Supp}(\phi) = \overline{\phi^{-1}(\mathbb{R} - \{0\})}$. Let $\{U_1, \dots, U_n\}$ be a finite open covering of the normal space X. An indexed family of continuous functions $\phi_i: X \to [0,1]$ is said to be a **partition of unity** dominated by $\{U_i\}$ if

1. $\operatorname{Supp}(\phi_i) \subseteq U_i$ for each i
2. $\sum_{i=1}^{n} \phi_i(x) = 1$ for each x

<u>Pf. of existence.</u> We can shrink $\{U_i\}$ to an open covering $\{V_i\}$ of X such that $\bar{V}_i \subseteq U_i$, by normality. (Define V_i inductively.) Shrink $\{V_i\}$ to $\{W_i\}$ by the same method. By Urysohn's Lemma choose $\psi_i: X \to [0,1]$ so that $\psi_i(\bar{W}_i) = \{1\}$ and $\psi_i(X - V_i) = \{0\}$. Scale to get $\phi_i(x) = \frac{\psi_j(x)}{\sum_{i=1}^{n} \psi_i(x)}$.

If X is a compact m-manifold, then X can be imbedded in $\mathbb{R}^N$ for some N.
<u>Pf.</u> Cover X by finitely many open sets $\{U_i\}$, where U_i can be imbedded in $\mathbb{R}^m$ via g_i. Let $\phi_1, \dots, \phi_n$ be a partition of unity dominated by U_i. Let

$$h_i(x) = \begin{cases} \phi_i(x)\vec{g}_i(x) \text{ for } x \in U_i \\ \vec{0} \text{ for } x \in X - A_i \end{cases}$$

Define $F: X \to ((\mathbb{R})^n \times (\mathbb{R}^m)^n)$ by $F(x) = \big(\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x)\big)$. F is injective ($g_i$ is injective on U_i; for each x some $\phi_i(x)$ is positive) so this works.

5-3	Dimension Theory

Dimension Theory

A collection of subsets of X has **order** n if some point of X lies in n elements of $\mathcal{A}$ and no point of X lies in more than n elements of $\mathcal{A}$.
X is finite dimensional if there is some integer m such that for every open cover $\mathcal{A}$ of X, there is an open cover $\mathcal{B}$ of X refining A with order at most $m+1$. The **topological dimension** $\dim(X)$ of X is the smallest value of m for which this statement holds.

If Y is a closed subspace of X then $\dim(Y) \le \dim(X)$.
If $X = Y \cup Z$ where Y, Z are closed, then $\dim(X) = \max\{\dim(Y), \dim(Z)\}$.
Pf.

1. If $\mathcal{A}$ is an open cover, there is an open cover refining $\mathcal{A}$ and has order at most $m+1 = \max\{\dim(Y), \dim(Z)\} + 1$ at points of Y. Consider $\{A \cap Y | A \in \mathcal{A}\}$; take a open cover refinement $\mathcal{B}$. For $B \in \mathcal{B}$, choose open U_B so $U_B \cap Y = B$ and choose $A_B \in \mathcal{A}$ so $B \subseteq A_B$; take the cover $\mathcal{C} = \{U_B \cap A_B\}$.
2. Let $\mathcal{C}'$ be a refinement of $\mathcal{C}$ with order at most $m+1$ at points of Z. Define $f: \mathcal{C}' \to \mathcal{C}$ so that $C \subseteq f(C) \in \mathcal{B}$. Let $D(B) = \bigcup_{C \in \mathcal{C}, f(C) = B} C$. Take $\mathcal{D} = \{D(B)\}$.

Every compact subspace of $\mathbb{R}^N$ has topological dimension N.
Pf. (that it's $\le N$)

1. Break $\mathbb{R}^N$ into unit cubes. Let $\mathcal{J} = \{(n, n+1) | n \in \mathbb{Z}\}$ and $\mathcal{K} = \{\{n\} | n \in \mathbb{Z}\}$. A M-cube is in the form $A_1 \times \cdots \times A_N$ where exactly M of the sets are in $\mathcal{J}$ and the rest are in $\mathcal{K}$. Expand each cube into an open set so for given M, no two expanded M-cubes intersect. This is a open cover of order at most m+1.
2. Given an open covering $\{A_n\}$ of compact subspace X, shrink the cover above so unit cubes become $\frac{\delta}{N}$-cubes, where δ is a Lebesgue number of X (a number such that every subset with diameter less than $\frac{\delta}{N}$ inside X entirely inside some A_n), and intersect the sets in the two covers.

Points $\{v_0, \ldots, v_k\}$ in $\mathbb{R}^N$ are **affinely independent** if $\sum_{i=0}^{k} a_i v_i = 0, \sum_{i=1}^{k} a_i = 0$ imply each $a_i = 0$. A set A of points is in general position in $\mathbb{R}^N$ if every subset containing at most $N+1$ points is affinely independent.

<u>Imbedding Theorem:</u> Every compact metrizable space X of topological dimension m can be imbedded in $\mathbb{R}^{2m+1}$. (Ex. Graphs can be imbedded in $\mathbb{R}^3$.)
Pf.

1. Use the square metric on $\mathbb{R}^{2m+1}$: $|\mathrm{x} - \mathrm{y}| = \max\{|x_i - y_i|\}$. Use the sup metric on $\mathcal{C}(X, \mathbb{R}^N)$: $\rho(f, g) = \sup\{|f(x) - g(x)|; x \in X\}$. Given continuous $f: X \to \mathbb{R}^{2m+1}$, define $\Delta(f) = \sup\{\operatorname{diam} f^{-1}(\{z\}) | z \in f(X)\}$ (how far f deviates from being injective). Define $U_\varepsilon = \{f \in \mathcal{C}(X, \mathbb{R}^{2m+1}) | \Delta(f) < \varepsilon\}$.
2. U_ε is open and dense. To show it's open, given $f \in U_\varepsilon$, bound $|f(x) - f(y)|$ on $A = \{(x, y) | d(x, y) \ge b\}$. To show it's dense, given $f \in \mathcal{C}(X, \mathbb{R}^{2m+1})$, cover X with finitely many open sets $\{U_i\}$ so that
    a. $\operatorname{diam}(U_i) < \frac{\varepsilon}{2}$ in X
    b. $\operatorname{diam} f(U_i) < \frac{\delta}{2}$ in $\mathbb{R}^{2m+1}$
    c. $\{U_i\}$ has order at most $m+1$.

   Take a partition of unity ϕ_i dominated by $\{U_i\}$. For each i choose $x_i \in U_i$ and a point $\mathrm{z}_i \in \mathbb{R}^{2m+1}$ within $\frac{\delta}{2}$ of the point $f(x_i)$ such that $\{\mathrm{z}_1, \ldots, \mathrm{z}_n\}$ is in general position in

$\mathbb{R}^{2m+1}$ (possible by induction and fact that $\mathbb{R}^{2m+1}$ is Baire space). Define $g(x) = \sum_i^n \phi_i(x)\mathrm{z}_i$. We have $\rho(f,g) < \delta$ and $g \in U_\varepsilon$: If $g(x) = g(y)$ then $\sum_{i=1}^n [\phi_i(x) - \phi_i(y)]\mathrm{z}_i = 0$. Since $\{U_i\}$ has order at most $m+1$, at most $2(m+1)$ of the coefficients are nonzero. Since z_i are in general position in $\mathbb{R}^{2m+1}$, this forces all coefficients to be 0. Then $y \in U_i$.

3. By Baire, $\bigcap_{n=1}^\infty U_{1/n}$ is dense in $\mathcal{C}(X, \mathbb{R}^{2m+1})$ and nonempty. Any function in this intersection gives an imbedding.

Cor. Every compact m-manifold has dimension equal to m so can be imbedded in $\mathbb{R}^{2m+1}$. If X is compact metrizable, X can be imbedded in some $\mathbb{R}^N$ iff X has finite topological dimension.