Box (a.k.a Pigeonhole) Principle, Coloring, and Related Ideas

ERHS Math Club

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The Box/ Pigeonhole Principle: If $kn + 1$ objects are put in n groups, then one group will contain at least $k + 1$ objects.

1 Easy

1. In a drawer are 3 pairs of socks in each of the seven colors of the rainbow. How many socks do I need to take to be sure there is a triplet of matching socks for my three-legged pet alien?
2. Prove that in any set of n+1 integers there exist 2 whose difference is divisible by n.
3. (AHSME 1993) Label n disks "n" for $1 \leq n \leq 50$. Find the least number of disks that must be drawn to guarantee drawing at least ten disks with the same label.
4. (AHSME 1991) A circular table has 60 chairs around it. There are N people seated at the table in such a way that the next person to be seated must sit next to someone. Find the least possible N.
5. A computer has been used for 99 hours over a period of 12 days, A whole number of hours every day. Prove that on some pair of consecutive days, the computer was used at least 17 hours.
6. Five lattice points (points with integer coordinates) are chosen in the plane. Prove you can choose 2 points such that the line segment joining them passes through another lattice point.
7. n people meet in a room. Some shake hands. Prove that 2 people will have shaken the same number of hands.
8. Prove that for any n, there is a Fibonacci number that has n as a factor.
9. Prove that of any 10 points chosen within an equilateral triangle of side 1 there are 2 whose distance apart is at most 1/3.

2 Medium

1. (Dirichlet and Kronecker) Let a be irrational and p be a positive integer. a) There exist positive integers m, n such that $|ma - p| \leq \frac{1}{p}$. b) A man with step length a walks around a planet with circumference 1. The circle has a ditch of width $\epsilon > 0$. Prove that sooner or later he will step into the ditch.
2. Let S be a set of n integers. Prove that S contains a subset such that the sum of its elements is divisible by n. (Hint: Partial sums)
3. (1973 NYSML) A set S of distinct integers each of which is greater than or equal to 1 and less than or equal to 100 is given. (1- easy) If S consists of 51 elements, is it possible that no element of S is the sum of two distinct elements of S? (2- medium) If S consists of 52 elements, prove that the largest element of S is the sum of 2 distinct elements of S and the smallest element is the difference of two distinct elements of S. (3- hard) If S consists of 69 elements prove that at least 1 element of S is the sum of 3 distinct elements of S.
4. (APMO) Does the set 1,2,...3000 contain a subset A of 2000 elements such that x is in A implies $2x$ is not in A?

5. Each point of the plane is colored white, black, or red. Prove that it is possible to find 2 points which are painted in the same color and the distance between them is 1.
6. 8 points are inside or on a circle of radius 1. Prove that 2 of them are less than 1 unit apart.
7. Each of 10 segments is longer than 1 but shorter than 55. Prove you can select 3 segments that form a triangle.
8. Inside the unit square lie several circles with sum of circumferences equal to 10. Prove there exist infinitely many lines each of which intersects at least 4 circles.
9. Let S be a set of 25 points such that in any 3-element subset of S there exist two points with distance less than 1. Prove that there exists a 13-subset of S which can be covered by a circle of radius 1. (Hint: take the pair of points with MAXIMUM distance apart)
10. (BAMO 2006) The points of the plane are colored in black and white so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color. (Note this isn't a box principle problem)

3 Hard

1. The politicians in Congress were divided into 12 factions. After their first meeting they regrouped into 16 factions. (Each politician can only be in one faction at a time.) Prove that now at least 5 politicians are in smaller factions than before the meeting.
2. A sloppy tailor uses a machine to cut 120 square patches of area 1 from a 25x20 rectangle sheet. Prove that he can still cut a circular patch of diameter 1 from the remaining fabric. (Hint: where can the center of the circle NOT be, relative to the cut-out patches?)
3. Given a square grid S containing 49 points in 7 rows and 7 columns, a subset consisting of k points is selected. Find the maximum k such that no 4 points of T determine a rectangle having sides parallel to those of S.
4. Let S be a convex set in the plane that contains three noncollinear points. The points of S are colored by $p > 1$ colors. Prove that for any $n \geq 3$ there exist infinitely many congruent n-gons whose vertices are colored by the same color.
5. Let n be a given positive integer. Consider a set S of n points, with no 3 collinear, such that the distance between any pair of points in the set is least 1. We define the radius of the set, denoted by r_S, as the largest circumradius of the triangles with their vertices in S. Determine the minimum value of r_S. (Hint: extended law of sines: $2R = \frac{a}{\sin A}$)
6. There are 650 points inside a circle of radius 16. Prove there exists a ring with inner radius 2 and outer radius 3 covering 10 of these points.
7. (ISL 1999) Prove that the positive integers cannot be partitioned into 3 nonempty sets such that if x is in one set and y is in another, then $x^2 + xy + y^2$ is in the third.
8. (Romania 1996) Let n be an integer greater than 2 and let S be a $3n^2$ - element subset of the set $\{1, 2, \ldots, n^3\}$. Prove that one can find nine distinct numbers $a_1, a_2, \ldots, a_9$ in S such that the system

$$a_1x + a_2y + a_3z = 0$$
$$a_4x + a_5y + a_6z = 0$$
$$a_7x + a_8y + a_9z = 0$$

has a solution (x_0, y_0, z_0) in nonzero integers.