2010 AwesomeMath UTD Team Contest Part 2

1 Algebra

1. Let $n \geq 2$. How many polynomials $Q(x)$ with real coefficients of degree at most $n-1$ are there such that

$$x(x-1)\cdots(x-n)Q(x)+x^2+1$$

is the square of a polynomial?

Solution The given condition says

$$f(x)^2 = x(x-1)\cdots(x-n)Q(x)+x^2+1 \tag{1}$$

for some polynomial $f(x)$ of degree at most n. Plugging $x = 0, 1, \ldots, n$ into (1) gives

$$f(x) = \pm\sqrt{x^2+1}, \text{ when } x = 0, 1, \ldots, n. \tag{2}$$

The following is key:

Fact 1. Given $n+1$ points $(x_0, y_0), \ldots, (x_n, y_n)$ with distinct x-coordinates, there exists exactly one polynomial f of degree at most n so that $f(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

Applying this to (2) we get 2^{n+1} possibilities for $f(x)$ since we have 2 choices of sign for each of $x = 0, 1, \ldots, n$. If $f(x)$ is a solution to (2) then so is $-f(x)$; we get 2^n possibilities for $f(x)^2$. Solve (1) to get 2^n possibilities for $Q(x)$:

$$Q(x) = \frac{f(x)^2 - x^2 - 1}{x(x-1)\cdots(x-n)}$$

Each such polynomial is a valid solution because $f(x)^2 - x^2 - 1$ is zero at $x = 0, 1, \ldots, n$ and hence is divisible by $x(x-1)\cdots(x-n)$. ■

2. Let $a_1, \ldots, a_5$ be real numbers and x, y be real numbers such that

$$(a_1+a_2+a_3+a_4-a_5)^2 \geq 3(a_1^2+a_2^2+a_3^2+a_4^2-a_5^2).$$

Prove that

$$(a_1+a_2+a_3+a_4-a_5-x-y)^2 \geq a_1^2+a_2^2+a_3^2+a_4^2-a_5^2-x^2-y^2.$$

Solution By T_2's Lemma (or Cauchy-Schwarz, or QM-AM),

$$\begin{aligned}\frac{(a_1+a_2+a_3+a_4-a_5-x-y)^2}{1}+\frac{x^2}{1}+\frac{y^2}{1} &\geq \frac{(a_1+a_2+a_3+a_4-a_5)^2}{3}\\ &\geq a_1^2+a_2^2+a_3^2+a_4^2-a_5^2.\end{aligned}$$

This rearranges to the desired inequality.

3. $n > 1$ bunnies sit on a number line such that the maximum distance between any two of them is d. Each step, 2 bunnies are selected. The bunny at the left, say A, jumps some distance $x > 0$ to the right, while the bunny at the right, say B, jumps the same distance x to the left, such that A is still to the left of (or occupies the same location as) B. (Bunnies may jump over each other.) After a finite number of steps, let S be the sum of the distances traveled by all bunnies. Let $L(n, d)$ be the smallest number so that we always have $S \le L(n, d)$. Find $L(n, d)$.

   Solution Number the bunnies $1, 2, \ldots, n$, and let their positions be $x_1, \ldots, x_n$, respectively, where $0 \le x_1, \ldots, x_n \le d$. Define

$$f(x_1, \ldots, x_n) = \sum_{1 \le i < j \le n} |x_i - x_j|. \tag{3}$$

   Lemma 1. After the bunnies travel a distance of l, then $f(x_1, \ldots, x_n)$ decreases by at least l. Equality holds iff the only time bunnies move jump toward each other is when there are no other bunnies in between.

   Proof. Decomposing each step into substeps as necessary, we may assume that no bunny crosses another bunny during a step. It suffices to prove the claim for a step. WLOG, assume the bunnies are in increasing order from left to right. Suppose bunnies i and j at positions $x_i < x_j$ each move a distance of l during this step. Terms in (3) not involving x_i, x_j do not change. Now, for $k < i$ or $k > j$, we can pair the terms $|x_k - x_i| + |x_k - x_j|$; one increases by l while the other term decreases by l. If $i < k < j$, then both terms $|x_k - x_i| + |x_k - x_j|$ decrease by l. The term $|x_j - x_i|$ decreases by $2l$, so the whole sum decreases by at least $2l$, with equality iff there is no bunny between bunnies i and j. □

   Hence no the total distance the bunnies move cannot be greater than $f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are the initial positions. The function $f(x_1, \ldots, x_n)$ achieves a maximum on the closed and bounded set $0 \le x_1, \ldots, x_n \le d$; since the functions $|x - a|$ are all convex, by looking at f in each variable separately, we get that the maximum must be obtained when each variable $x_1, \ldots, x_n$ is equal to 0 or d. If k of them are equal to 0, then

$$f(x_1, \ldots, x_n) = k(n - k)d,$$

   which attains maximum for the value of k closest to the vertex $\frac{n}{2}$ of the quadratic $x(n - x)$. When n is even the maximum is $\frac{n^2 d}{4}$; when n is odd the maximum is $\frac{(n-1)(n+1)d}{4}$. Now we show these are indeed the values of $L(n, d)$.

   Lemma 2. Given any arrangement of bunnies in increasing order from left to right, if we repeatedly move pairs of bunnies $(1, 2), (2, 3), \ldots, (n-1, n)$, each time moving the 2 bunnies in the pair to the same location, in that order, then the location of all bunnies will converge to $\bar{x} = \dfrac{\sum_{i=1}^n x_i}{n}$.

Proof. Since the sum of the positions of the bunnies is invariant, if the position of all bunnies converges to x then we have $nx = x_1 + \ldots + x_n \Rightarrow x = \bar{x}$. Let $x_i(t)$ be the position of the ith bunny after t repetitions of the algorithm above. Then $x_n(t) - x_1(t)$ is decreasing, so it converges to some $D \geq 0$. It suffices to consider $D \neq 0$. Note that after the $(t+1)$th step of the algorithm, every point between $x_1(t)$ and $x_n(t)$ has been jumped over (or on) by one bunny. Indeed, after moving pairs $(1, 2), \ldots, (i-1, i)$ to the same location, bunny i has not moved right. Then when $(i, i+1)$ are moved together, all points between $x_i(t)$ and $x_{i+1}(t)$ have been jumped over (or on). So the total distance traveled by the bunnies in the $(t+1)$th step is at least $x_n(t) - x_1(t) \geq D$, and the total distance traveled by the bunnies is unbounded if the step is repeated, a contradiction. □

Now if n is even, then take $x_1 = \cdots = x_{\frac{n}{2}} = 0$ and $x_{\frac{n}{2}+1} = \cdots = x_n = d$, and if n is odd, take $x_1 = \cdots = x_{\frac{n+1}{2}} = 0$, $x_{\frac{n+3}{2}} = \cdots = x_n = d$, and carry out the algorithm described in the lemma. Then by Lemma (1), in the limit all the bunnies travel a total distance of $\frac{n^2 d}{4}$ or $\frac{(n^2-1)d}{4}$, respectively, because we have that f attains this value, and no bunny jumps over another bunny. Hence

$$L(n,d) = \begin{cases} \frac{n^2 d}{4}, n \text{ even} \\ \frac{(n^2-1)d}{4}, n \text{ odd} \end{cases} \tag{4}$$

■

2 Combinatorics

1. Temporarily withheld.

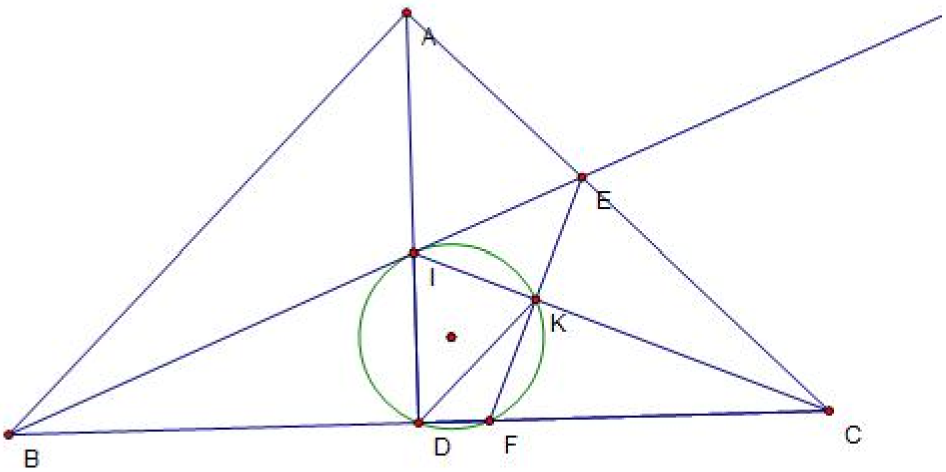
2. Temporarily withheld.

3. Temporarily withheld.

3 Geometry

1. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

   Solution $60°$ or $90°$.



The angle bisectors AD and BE intersect at the incenter I of ΔABC. Reflect E across CI to point F on BC. Then $EF \perp CI$ and $\angle IFK = \angle IEK = 45°$. Since K is the incenter of ΔADC, DK bisects $\angle ADC$, and $\angle IDK = 45° = \angle IFK$. Hence I, K, D, F are concyclic.

If $F = D$ then $CE = CF = CD$. Since $AD \perp BC$ and BE, AC are the reflections of AD, BC across CI, we get $BE \perp AC$ and $AB = AC$, i.e. ΔABC is equilateral and $\angle CAB = 60°$.

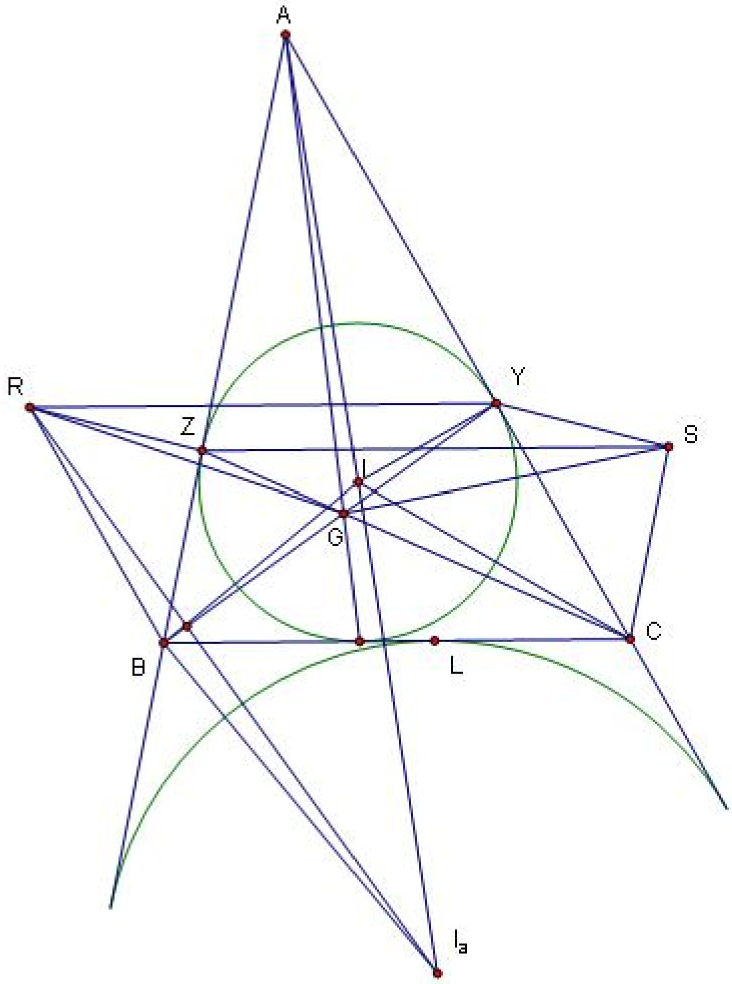
Else, $\angle IKE = \angle IKF = \angle IDF = 90°$, and

$$\begin{aligned}\angle A = \angle BEC - \angle EBA &= 45° + \angle KEC - \angle EBA \\ &= 45° + \angle EKI - \angle ECI - \angle EBA = 45° + 90° - (90 - \angle A/2)\end{aligned}$$

and $\angle A = 90°$. ■

2. Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

Solution



Let the excircle ω_A opposite A intersect AB and AC at D and E, respectively, let a, b, c be the side lengths of the triangle and let s be the semiperimeter. Now,

$$BR = CY = s - c = BD,$$

so B has equal power with respect to ω_A and the point (degenerate circle) R. We have

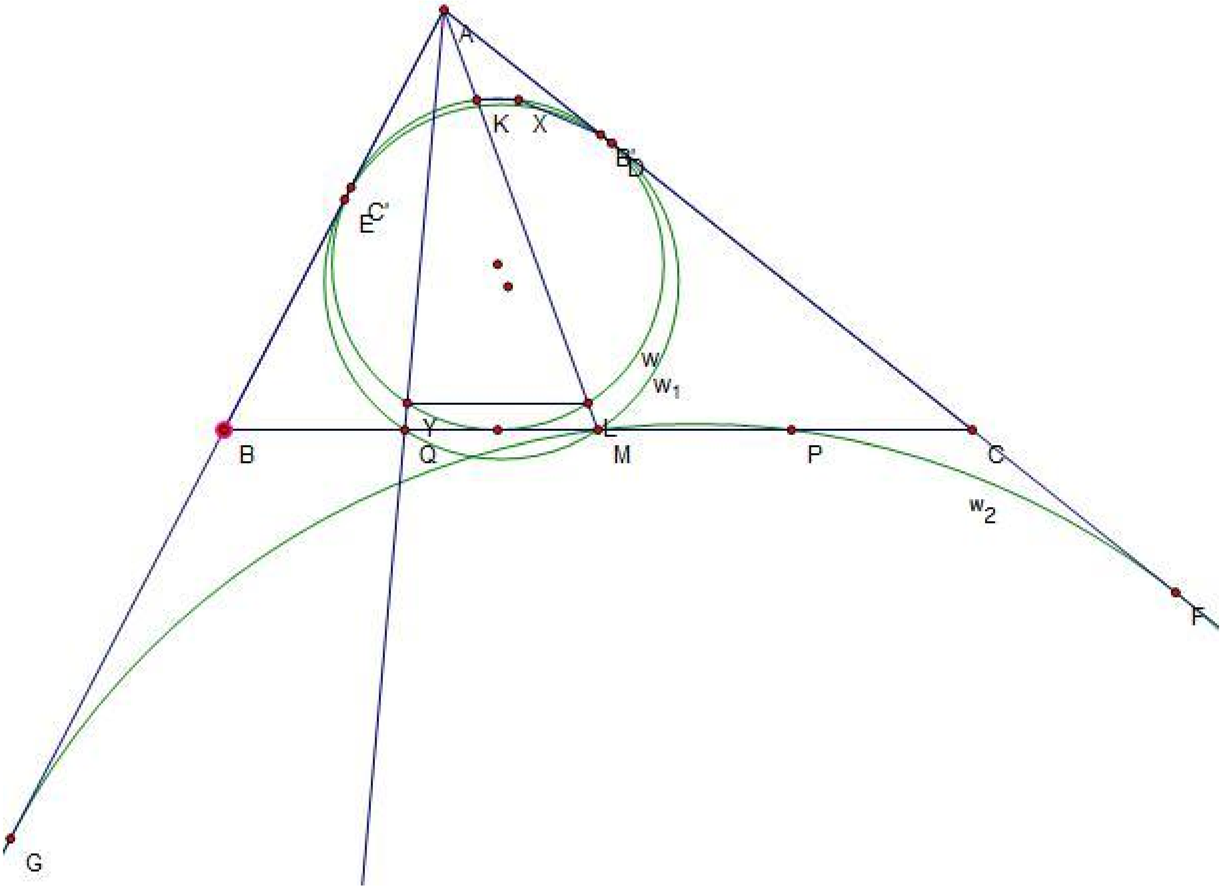
$$YR = CB = (s - b) + (s - c) = CY + CE = YE$$

so Y has equal power with respect to ω_A and R. Hence BY is the radical axis of R and ω_A. Similarly, CZ is the radical axis of S and ω_A. Hence their point of intersection G is the radical center of R, Y, and ω_A. Hence G has equal power with respect to R and S, and $GR = GS$. ■

3. The median AM of ΔABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

   Solution Without loss of generality, $AC > AB$.

   Let ω intersect AB and AC at C' and B', respectively. Since $YL \parallel QM$, there is a homothety H_1 centered at A sending YL to QM. H_1 sends ω to a circle ω_1 tangent to AB and AC. Suppose ω_1 intersects AB and AC at E and D. Similarly, a homothety H_2 centered at A sends KX to MP, and ω to a circle ω_2 tangent to AB and AC, say at G and F.



By equal tangents,

$$EG = AG - AE = AF - AD = DF. \tag{5}$$

By power-of-a-point with respect of B and C with respect to ω_1 and ω_2, setting $BQ = x, CP = y, BC = a$ we get

$$\begin{aligned} BE^2 &= BQ \cdot BM = x \cdot \frac{a}{2} \\ BG^2 &= BM \cdot BP = \frac{a}{2} \cdot (a - y) \\ CD^2 &= CM \cdot CQ = \frac{a}{2} \cdot (a - x) \\ CF^2 &= CP \cdot CM = y \cdot \frac{a}{2} \end{aligned}$$

From (5), we get $BE + BG = CD + CF$. Substituting the above into this equation and dividing by $\sqrt{a/2}$,

$$\begin{aligned} \sqrt{x} + \sqrt{a-y} &= \sqrt{a-x} + \sqrt{y} \\ \sqrt{x} - \sqrt{a-x} &= \sqrt{y} - \sqrt{a-y} \end{aligned}$$

Since $\sqrt{x} - \sqrt{a-x}$ is an increasing function, $x = y, BQ = CP, BP = CQ$. ■

4 Number Theory

1. There are $n \geq 51$ points in the plane with integer coordinates, such that the distance between any two is an integer. Prove that at least 49 percent of the distances are even.

   Solution Let the points be $(x_1, y_1), \ldots, (x_n, y_n)$. We show that either all the x_i have the same parity, or all the y_i have the same parity.

   Suppose that the x_i do not all have the same parity; suppose $x_i \not\equiv x_j \pmod 2$. Then the square of the distance between (x_i, y_i) and (x_j, y_j) is $(x_i - x_j)^2 + (y_i - y_j)^2$. If $y_i \not\equiv y_j \pmod 2$, then $(x_i - x_j)^2 + (y_i - y_j)^2 \equiv 1 + 1 \equiv 2 \pmod 4$ so cannot be a square. Hence $y_i \equiv y_j \pmod 2$. If $x_i \equiv x_k \pmod 2$ then finding $x_j \not\equiv x_i \pmod 2$ we get $y_i \equiv y_j \equiv y_k \pmod 2$ as well. Thus all the y_i have the same parity.

   Without loss of generality, all the y_i have the same parity. Suppose k of the x_i them are even; then $n - k$ of them are odd. (x_i, y_i) and (x_j, y_j) have even distance apart if $x_i - x_j$ and $y_i - y_j$ are both even. There are

   $$\frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} = \frac{k^2 + (n-k)^2 - n}{2}$$

   such pairs. The proportion of even distances is at least

   $$\frac{[k^2 + (n-k)^2 - n]/2}{n(n-1)/2} \geq \frac{\left(\frac{n}{2}\right)^2 - n}{n(n-1)} = \frac{\frac{n}{2} - 1}{n-1} \geq \frac{\frac{51}{2} - 1}{51 - 1} = \frac{49}{100}.$$

   ■

2. Let p, q, r be distinct primes such that

   $$pq \mid r^p + r^q.$$

   Prove that either p or q equals 2.

   Solution Suppose the relation holds but $p \neq 2, q \neq 2$. By Fermat's Little Theorem, $r^p \equiv r \pmod p$ and $r^q \equiv r \pmod q$. Then since r is relatively prime to p, q,

   $$\begin{aligned} r^p + r^q \equiv 0 \pmod p &\implies \\ r^{q-1} \equiv -1 \pmod p& \\ r^p + r^q \equiv 0 \pmod q &\implies \\ r^{p-1} \equiv -1 \pmod q& \end{aligned}$$

   Since $-1 \not\equiv 1 \pmod{p, q}$, we get

   $$\text{ord}_p(r) \nmid q - 1, \text{ord}_q(r) \nmid p - 1. \qquad (6)$$

   Since

   $$\begin{aligned} r^{2(q-1)} &\equiv 1 \pmod p \\ r^{2(p-1)} &\equiv 1 \pmod q, \end{aligned}$$

we get

$$\mathrm{ord}_p(r) \mid 2(q-1), \mathrm{ord}_q(r) \mid 2(p-1). \tag{7}$$

For an integer n let $v_2(n)$ denote the highest power of 2 dividing n. Let $x = v_2(\mathrm{ord}_p(r))$ and $y = v_2(\mathrm{ord}_q(r))$. From relations in (6) and (7),

$$\begin{aligned} x &= v_2(2(q-1)) = v_2(q-1) + 1 \\ y &= v_2(p-1) + 1. \end{aligned} \tag{8}$$

By Fermat's Little Theorem, $\mathrm{ord}_p(r) \mid p-1$ and $\mathrm{ord}_q(r) \mid q-1$. Hence

$$\begin{aligned} x &\le v_2(p-1) \\ y &\le v_2(q-1) \end{aligned} \tag{9}$$

Putting (8) and (9) together, we get $x \le y-1, y \le x-1$, contradiction. ■

3. Find all solutions in positive integers to $a^2 + 2b^2 = (x^5 - x^3 - 1)(x^5 - x^3 - 3)$. (Hint: You may use the fact that -2 is a quadratic residue modulo a prime p if and only if $p \equiv 2, 1$, or $3 \pmod 8$.)

Solution

Lemma 3. Let a, b be integers. If p is an $8k+5$ or $8k+7$ prime, then the highest power of p dividing $a^2 + 2b^2$ is an even power.

Proof. Suppose $a^2 + 2b^2 \equiv 0 \pmod p$. Let p^r be the highest power of p dividing $a^2 + 2b^2$. Let $a' = \frac{a}{p^r}$ and $b' = \frac{b}{p^r}$; then and a', b' are not both divisible by p. We have that $a'^2 + 2b'^2 = \frac{a^2+2b^2}{p^{2r}}$; it suffices to show that $p \nmid a'^2 + 2b'^2$. Suppose by way of contradiction that $a'^2 + 2b'^2 \equiv 0 \pmod p$. Since a' and b' are not both divisible by p, neither is. We get $(a'b'^{-1})^2 \equiv -2 \pmod p$, contradicting the fact that -2 is not a quadratic residue modulo p. □

Note $a = b = x = 1$ is a solution. We show this is the only solution. If $x > 1$, then both $x^5 - x^3 - 1, x^5 - x^3 - 3 > 0$. Now $x^5 - x^3 \equiv 0 \pmod 8$ so

$$\begin{aligned} x^5 - x^3 - 1 &\equiv 7 \pmod 8 \\ x^5 - x^3 - 3 &\equiv 5 \pmod 8 \end{aligned}$$

However, the product of an even number of $8k+5, 8k+7$ primes and an arbitrary number of $8k+1, 8k+3$ primes cannot be congruent to 5 or 7 modulo 8. Moreover, $x^5 - x^3 - 1$ and $x^5 - x^3 - 3$ are relatively prime since their difference is 2 and both are odd. Hence they have no common prime factor. It follows that $(x^5 - x^3 - 1)(x^5 - x^3 - 3)$ has a $8k+5$ or $8k+7$ prime appearing an odd number of times, and cannot equal $a^2 + 2b^2$. ■

5 Problem Credits

A1 Holden Lee

A2 Romania 1983

A3 Holden Lee

C1 Temporarily withheld.

C2 Temporarily withheld.

C3 Temporarily withheld.

G1 IMO 2009/4

G2 IMO Shortlist, 2009/G3 (Figure due to livetolove212)

G3 IMO Shortlist, 2005/G6

N1 Andrei Ciupan

N2 Based off many past Olympiad problems of the same flavor.

N3 Holden Lee

And... Holden Lee for LaTeXing up everything!