Team Contest Round 2

AwesomeMath

July 20-26, 2010

All answers must be proved!

1 Algebra

1. Let $n \geq 2$. How many polynomials $Q(x)$ of degree at most $n-1$ are there such that
$$x(x-1)\cdots(x-n)Q(x)+x^2+1$$
is the square of a polynomial?

2. Let $a_1,\ldots,a_5$ be real numbers and x,y be real numbers such that
$$(a_1+a_2+a_3+a_4-a_5)^2 \geq 3(a_1^2+a_2^2+a_3^2+a_4^2-a_5^2).$$
Prove that
$$(a_1+a_2+a_3+a_4-a_5-x-y)^2 \geq a_1^2+a_2^2+a_3^2+a_4^2-a_5^2-x^2-y^2.$$

3. $n>1$ bunnies sit on a number line such that the maximum distance between any two of them is d. Each step, 2 bunnies are selected. The bunny at the left, say A, jumps some distance $x>0$ to the right, while the bunny at the right, say B, jumps the same distance x to the left, such that A is still to the left of (or occupies the same location as) B. (Bunnies may jump over each other.) After a finite number of steps, let S be the sum of the distances traveled by all bunnies. Let $L(n,d)$ be the smallest number so that we always have $S \leq L(n,d)$. Find $L(n,d)$.

2 Combinatorics

1. Let S be a set of n positive integers, and let m be a positive integer. Prove that there are at least 2^{n-m+1} subsets of S with sum of elements divisible by m. Include the empty set in your count.

2. In a magic trick, the audience arranges n coins in a row, choosing whether they are heads or tails. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and Brian Basham is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, Brian tries to determine the number chosen by the audience. Find all n for which this is possible.

3. Let $f(n, r)$ denote the maximum number of edges a graph with n vertices can have so that it does not contain a complete bipartite subgraph $K_{r,r}$. Prove that

$$f(n, r) \leq cn^{2-1/r}$$

for some constant c depending only on r.

3 Geometry

1. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

2. Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

3. The median AM of ΔABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

4 Number Theory

1. There are $n \geq 51$ points in the plane with integer coordinates, such that the distance between any two is an integer. Prove that at least 49 percent of the distances are even.

2. Let p, q, r be distinct primes such that

$$pq \mid r^p + r^q.$$

Prove that either p or q equals 2.

3. Find all solutions in positive integers to $a^2 + 2b^2 = (x^5 - x^3 - 1)(x^5 - x^3 - 3)$. (Hint: You may use the fact that -2 is a quadratic residue modulo a prime p if and only if $p \equiv 2, 1, \text{ or } 3 \pmod 8$.)