Team Contest Round 1

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All answers must be proved!

1 Algebra

1. Prove that for all real a, b, c,

$$2^{-2/3}(\max(a,b,c) - \min(a,b,c)) \geq \sqrt[3]{|a-b||b-c||a-c|}.$$

2. Does there exist a nonlinear function f from the nonnegative reals to the nonnegative reals so that

$$\min_{0<x<t}[f(x) + f(t-x)] \leq f(t) \leq \max_{0<x<t}[f(x) + f(t-x)]$$

   for all positive t?

3. Let $x, y > 0$. Prove that

$$\frac{18}{(x+y)^4} \leq \frac{2}{(x-y)^4} + \frac{1}{x^3y + y^3x}$$

   and find when equality holds.

4. 7 points $Q_1, \ldots, Q_7$ are equally spaced on a circle of radius 1 centered at O. Point P is on ray OQ_7 so that $OP = 2$. Find the product

$$\prod_{k=1}^{7} PQ_i.$$

5. Find all polynomials $p(x) = a_n x^n + \cdots + a_1 x + a_0$ satisfying the following:

   (a) $\{a_n, a_{n-1}, \ldots, a_1, a_0\} = \{0, 1, \ldots, n-1, n\}$, and $a_n \neq 0$.

   (b) $p(x)$ has only rational roots.

6. For a polynomial
$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
define
$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_n^2.$$
Let $g(x) = 3x^2 + 7x + 2$. Find $f(x) \in \mathbb{R}[x]$ such that
   (a) $f(0) = 1$ and
   (b) For all $n \geq 0$, $\Gamma(f(x)^n) = \Gamma(g(x)^n)$.

2 Combinatorics

1. Let n, m, k be positive integers such that $n \geq km$. Find the number of m-tuples of positive integers $(a_1, \ldots, a_m)$ so that
$$a_1 + \cdots + a_m = n$$
and $a_i \geq k$ for each i.

2. Pam has a list of numbers 1 through n, permuted randomly, in a row. She reads the numbers from left to right, circling the numbers 1, then 2, and so on. If she reaches the end of the list with numbers left uncircled, she starts reading from the beginning of the list again. What is the probability that she finishes circling all numbers the third time she reads the list (and not before)?

3. For positive integers $a_1, \ldots, a_{2010}$ such that $a_1 - a_2, a_2 - a_3, \ldots, a_{2009} - a_{2010}$ are all distinct, find the minimum possible number of distinct elements of the set $\{a_1, \ldots, a_{2010}\}$.

4. The students at AwesomeMath were divided into 18 teams for the team contest, each with an arbitrary but nonzero number of people. After a change in rules they regrouped into 12 teams. Prove that at least 7 students are in larger teams than before.

5. Let S be a set of n positive integers, and let m be a positive integer. Prove that there are at least 2^{n-m+1} subsets of S with sum of elements divisible by m. Include the empty set in your count.

6. Let $f(n, r)$ denote the maximum number of edges a graph with n vertices can have so that it does not contain a complete bipartite subgraph $K_{r,r}$. Prove that
$$f(n, r) \leq cn^{2-1/r}$$
for some constant c depending only on r.

3 Geometry

1. Let ABC be a triangle where the incircle touches BC, CA, AB at D, E, F. AD intersects the incircle again at P. If $PD = 6$, $PE = 3$, $PF = 2$, then find $\frac{DF \cdot DE}{PF \cdot PE}$.

2. Given points X, Y, Z, construct a triangle $\triangle ABC$ for which X is the circumcenter, Y is the midpoint of BC and Z is the foot of the altitude from B to AC.

3. For which numbers n can a square be cut into concave n-gons?

4. Let A be a fixed point and l a fixed line. P is a variable point on l. Q is a point on ray AP such that $AP \cdot AQ = k^2$ where k is some constant. Find the locus of Q.

5. A regular pentagon $ABCDE$ is dilated about anywhere with a dilation of positive magnitude and then rotated $36°$ to pentagon $A'B'C'D'E'$. Find the minimum value of $AA' + BB' + CC' + DD' + EE'$.

6. Perpendiculars from B, C meet the angle bisector of $\angle A$ in $\triangle ABC$ at P, Q, respectively. R is such that $PR \parallel AB$ and $QR \parallel AC$. If X is a point such that BX and CX are tangents to the circumcircle of $\triangle ABC$, then prove that A, R, X are collinear.

4 Number Theory

1. a, b, c, d, e are positive integers satisfying $a^3 + b^3 + c^3 + d^3 = e^3$. Find the largest positive integer n so that n is guaranteed to divide at least one of a, b, c, d, e.

2. Find all triplets of whole numbers (a, b, n) that satisfy $a^2 + b^2 = 2^n$.

3. Find all polynomials with integer coefficients $P(x)$ such that

$$\sum_{i=1}^{\infty} \frac{1}{2010^{P(i)}}$$

is rational.

4. Prove that for any positive integer k there exists an arithmetic sequence $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_k}{b_k}$ of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$ such that $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct and $\gcd(b_1, b_2, \ldots, b_k) = 1$.

5. Suppose $f(x)$ is a polynomial of degree d taking integer values such that

$$m - n \mid f(m) - f(n)$$

for all pairs of integers (m, n) satisfying $0 \le m, n \le d$. Is it necessarily true that

$$m - n \mid f(m) - f(n)$$

for all pairs of integers (m, n)?

6. Let n be a positive integer and p a prime satisfying $p > n^2 + 1$. Prove that for any nonzero residue m modulo p, there exist $a_1, \dots, a_n$, none of them equal to 0, satisfying

$$a_1^n + a_2^n + \cdots + a_n^n \equiv m \pmod{p}.$$