Team Contest Round 2

Alex, Brian, Holden, Juan, Michal, Tim

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All answers must be proved!

1 Algebra

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xyf(x+y) = (x+y)f(x)f(y), \qquad |f(x)| \geq x$$

for all $x, y \in \mathbb{R}$.

Solution Let $g(x) = \frac{f(x)}{x}$. The equation becomes $g(x+y) = g(x)g(y)$. $g(x)$ is positive for $x \neq 0$; we can let $h(x) = \ln g(x)$. The equation becomes $h(x+y) = h(x) + h(y)$. The second equation becomes $h(x) \geq 0$ for $x \geq 0$. By Cauchy's equation, $h(x) = cx$, $c \geq 1$. Then $f(x) = xc^x$.

2. Show that if a, b, c are positive and $a + b + c = 1$, then

$$(9abc+1)\left(\frac{1}{(1-a)^2} + \frac{1}{(1-b)^2} + \frac{1}{(1-c)^2}\right) \geq 9.$$

Solution Homogenize the inequality; we want to prove

$$(9abc + (a+b+c)^3)\left(\frac{1}{(b+c)^2} + \frac{1}{(a+c)^2} + \frac{1}{(a+b)^2}\right) \geq 9(a+b+c).$$

By Schur's inequality,

$$9abc + (a+b+c)^3 \geq 4(a+b+c)(ab+bc+ca)$$

so the inequality follows from Iran '96 (LOL!).

3. Let p be an odd prime and $n_1, \ldots, n_k$ be integers. Let

$$S = \left|\sum_{j=1}^{k} \cos\left(\frac{2\pi n_j}{p}\right)\right|.$$

Prove that either $S = 0$ or

$$S \geq k \left(\frac{1}{2k}\right)^{\frac{p-1}{2}}.$$

Solution (The high-tech solution) Let $\omega = e^{2\pi i/n}$; then $2\cos\left(\frac{2\pi j}{p}\right) = \omega^j + \omega^{-j}$. Since ω is an algebraic integer, so is $2\cos\left(\frac{2\pi j}{p}\right)$ for $0 < j < p$. Note $[\mathbb{Q}(\omega) : \mathbb{Q}(\omega + \overline{\omega})] = 2$ since ω satisfies a quadratic equation with coefficients in $\mathbb{Q}(\omega + \overline{\omega})$ (namely $x^2 - (\omega + \overline{\omega})x + 1 = 0$), and $\omega \notin \mathbb{R}$ implies $\omega \notin \mathbb{Q}(\omega + \overline{\omega})$. Since

$$n = [\mathbb{Q}(\omega) : \mathbb{Q}] = [\mathbb{Q}(\omega) : \mathbb{Q}(\omega + \overline{\omega})][\mathbb{Q}(\omega + \overline{\omega}) : \mathbb{Q}],$$

we get $[\mathbb{Q}(\omega + \overline{\omega}) : \mathbb{Q}] = \frac{n}{2}$. For $0 < j < p$, let φ_j denote the automorphism of $\mathbb{Q}(\omega)$ sending ω to ω^j. We have

$$G(\mathbb{Q}(\omega)/\mathbb{Q}) = \{\varphi^j \mid 0 < j < p\}.$$

Then the Galois group of $\mathbb{Q}(\omega + \overline{\omega})/\mathbb{Q}$ is

$$G(\mathbb{Q}(\omega)/\mathbb{Q}) = \left\{\varphi_m \mid 0 < m < \frac{p-1}{2}\right\}$$

since φ_m acts the same way as φ_{p-m}. Noting that $\varphi_m\left(\cos\left(\frac{2\pi j}{p}\right)\right) = \cos\left(\frac{2\pi jm}{p}\right)$,

$$P := \prod_{m=1}^{\frac{p-1}{2}} \varphi_m\left(\sum_{j=1}^{k} \cos\left(\frac{2\pi n_j}{p}\right)\right) = \prod_{m=1}^{\frac{p-1}{2}} \sum_{j=1}^{k} \cos\left(\frac{2\pi m n_j}{p}\right) \tag{1}$$

is invariant under all automorphisms in $G(\mathbb{Q}(\omega + \overline{\omega})/\mathbb{Q})$; hence by the Fixed Field Theorem, P is in the fixed field $\mathbb{Q}$. Since $2\sum_{j=1}^{k} \cos\left(\frac{2\pi n_j m}{p}\right)$ is an algebraic integer, so is the product over m, i.e. $2^{\frac{p-1}{2}}P$. Since the product is rational, it must be in $\mathbb{Z}$. None of the conjugates are 0, so $2^{\frac{p-1}{2}}P \geq 1$. Since each term in the sum is at most k, any term must have absolute value at least

$$\frac{1}{2^{\frac{p-1}{2}} k^{\frac{p-1}{2}-1}} = k\left(\frac{1}{2k}\right)^{\frac{p-1}{2}}.$$

(Low-tech solution) Note

$$\prod_{m=1}^{\frac{p-1}{2}} 2\sum_{j=1}^{k} \cos\left(\frac{2\pi m n_j}{p}\right) = \prod_{m=1}^{\frac{p-1}{2}} 2\sum_{j=1}^{k} (\omega^{n_j m} + \overline{\omega}^{n_j m})$$

can be written in the form $a_0 + a_1\omega + \cdots + a_{p-1}\omega^{p-1}$ for some integers a_i. The product stays the same if ω is replaced by ω^l for any $0 < l < p$ (as the factors are permuted), so we must have $a_1\omega^m + \cdots + a_{p-1}\omega^{m(p-1)}$ equal for any $0 < m < p$. (This step should be justified.) Thus for some c, $a_{p-1}x^{p-1} + \cdots + a_1 x + c = 0$ whenever x is a primitive nth root of unity. The irreducible polynomial of ω, namely $x^{p-1} + \cdots + x + 1$ must divide this polynomial, so $a_1 = \cdots = a_{p-1}$ and the product $a_0 + a_1\omega + \cdots + a_{p-1}\omega^{p-1}$ is an integer. Finish as above.

2 Combinatorics

1. There is a 9 by $5\sqrt{3}$ piece of paper with wrap around both on top/bottom and on the sides. What is the maximum number of points you can place so that no two are less than 1 away?

   Solution WHOOPS!

2. Fatty McButterpants is bored and decides to toss stones into two buckets. Both buckets start with no stones, and every second he throws a stone into the first bucket, the second bucket, or no buckets, each with probability $\frac{1}{3}$. After n seconds, what is the probability that the number of stones in each bucket is divisible by 3?

   Solution The answer is the sum of coefficients of $x^{3i}y^{3j}$ in the generating function $\frac{(1+x+y)^n}{3^n}$. Let ω be the third root of unity. The answer is

   $$\frac{\sum_{0\le i,j<3}(1+x+y)^n}{9\cdot 3^n} = \frac{3^n + 2(2+\omega)^n + 2(2+\omega^2)^n + (1+2\omega)^n + (1+2\omega^2)^n}{9\cdot 3^n}$$

3. Define a graph tiling of a graph G by a graph T to be a method of partitioning the edges of G into sets such that the edges of each set form a graph equivalent to T. We say G is tileable by T if there exists a graph tiling of G by T. Prove that for even a, K_n is tileable by $K_{a,a}$ if and only if $n = 2ka^2 + 1$ for some integer k.

   Solution Suppose that K_n is tileable by $K_{a,a}$. The edges coming out of a vertex must be partitioned into groups of a since each vertex in $K_{a,a}$ has degree a. Hence $a \mid n-1$. Next note the number of edges in $K_{a,a}$ must divide the number of edges in G, so $a^2 \mid \binom{n}{2} = \frac{n(n-1)}{2}$. Since $\gcd(n,a) = 1$ and n is odd, we must have $2a^2 \mid n-1$, proving the forward direction.

   Next we give a construction of a graph tiling for $n = 2ka^2+1$. Label the vertices of the graph from 0 to $n-1$. Consider the complete bipartite subgraphs G_{ij} of G with parts $\{i, i+1, \ldots, i+a-1\}$ and $\{i+ja^2, i+ja^2+a, \cdots, i+ja^2+a(a-1)\}$ (indices taken modulo n), for $0 \le i < n$ and $0 \le j < k$. There are kn such subgraphs, so in total these subgraphs have $ka^2n = \binom{n}{2}$ edges.

   Each edge is in one of these subgraphs: Take the edge from x to y. Without loss of generality, $(y-x) \bmod n \le ka^2$. Let i be the closest vertex before x (that is, in the opposite direction from y) such that $(y-i) \bmod n$ is a multiple of a. Then the edge between x and y is in G_{ij} where $j = \left\lfloor \frac{((y-i) \bmod n)-1}{a^2} \right\rfloor$. Since each edge is in one G_{ij}, each edge is in exactly on G_{ij}, and the G_{ij} give the desired graph tiling.

3 Geometry

1. Let ω_1 be a circle smaller than and internally tangent to ω_2 at T. A line l is tangent to ω_1 at T' and hits ω_2 at A and B. If $AT' = 7$ and $BT' = 56$, find the maximum possible area of $\triangle ATB$.

Solution Let TT' hit the circle again at M. By homothety M is the midpoint of arc $\widehat{AB}$. Hence TT' bisects $\angle ATB$. Then $TA/TB = T'A/T'B$. The locus of possible T given A, T', B is the Apollonius circle; it passes through T' and a point X on ray $\vec{BA}$ such that $XA = 9$. Now $XT' = 16$ so the radius of the circle is 8. The maximum area is

$$\frac{1}{2} \cdot 8 \cdot 63 = 252.$$

2. Let ω_1 and ω_2 be externally tangent at T and have centers O_1 and O_2. A common external tangent of these two circles intersects ω_1 at A_1 and ω_2 at A_2. Say P is such that TP is perpendicular to O_1O_2. Let PA_1 and PA_2 intersect ω_1 and ω_2 at B_1 and B_2. Prove that the circumcircle of $\triangle PB_1B_2$ is tangent to ω_1 and ω_2.

   Solution Let B_1B_2 hit ω_1 and ω_2 again at C_1 and C_2. It is easy to see that A_1, A_2, B_1, B_2 lie on a circle by power of a point. We can prove that $A_1B_1C_1$ is similar to PB_1B_2 using angle chasing. The homothety centered at B_1 takes $A_1B_1C_1$ to PB_2B_1 and thus takes ω_1 to the circumcircle of PB_1B_2. Thus the circles are tangent at B_1 and similarly at B_2.

3. Given two rays $\vec{XP}$ and $\vec{YP}$, a triangle ABC of fixed dimensions is placed with B on ray $\vec{XP}$ and C on ray $\vec{YP}$. Prove the locus of A lies on an ellipse.

   Solution Think of ABC as being fixed instead, and the rays as moving. By angles the locus of P, the intersection of the rays, while ABC is fixed is a circle. A rotation (depending on the angle) and a translation takes it back to the original locus.

4 Number Theory

1. For $n \in \mathbb{N}$, define

$$p_n = \prod_{d|n} d, \qquad q_n = \prod_{1 \le k \le n,\, \gcd(k,n)=1} k.$$

   Prove that there exists a sequence $x_1, x_2, \ldots$ such that both the following hold:

   (a) For every $m \in \mathbb{Z}$ there exists a unique i such that $x_i = m$.

   (b) For every $n \in \mathbb{N}$, $p_n^{q_n} \mid x_1 + \cdots + x_n$.

   Solution The only important fact is that p_n is relatively prime to p_{n+1}. Given any valid $x_1, \ldots, x_n$, we can make x_{n+2} anything we want by choosing x_{n+1} cleverly (with Chinese Remainder Theorem); moreover there are an infinite number of choices for x_{n+1}.

2. Suppose $f(x)$ is a polynomial of degree d that takes integer values for each integer x, and

$$m - n \mid f(m) - f(n)$$

   for all pairs of integers (m, n) satisfying $0 \le m, n \le d$. Is it necessarily true that

$$m - n \mid f(m) - f(n)$$

for all pairs of integers (m, n)?

Solution Yes.

Lemma 4.1: For $n \in \mathbb{N}_0$, let $l_n = \mathrm{lcm}(1, 2, \ldots, n)$ $(l_0 = 1)$.

$$m - n \mid l_i \left[\binom{m}{i} - \binom{n}{i} \right]$$

for all $m, n \in \mathbb{Z}, i \in \mathbb{N}_0$. (Note $\binom{x}{n}$ is defined as $\frac{x^{\underline{n}}}{n!}$.)

Proof. We induct on i. For $i = 0$ this is trivial. Suppose it true for $i - 1$. Write the RHS like this:

$$\frac{l_i}{i} \left[m\binom{m-1}{i-1} - n\binom{n-1}{i-1} \right] = \frac{l_i}{i} \left[m \left(\binom{m-1}{i-1} - \binom{n-1}{i-1} \right) + (m-n)\binom{n-1}{i-1} \right]$$

Since $\frac{l_i}{i}$ is an integer, by the induction hypothesis, $m - n$ divides this expression, finishing the induction step. □

Lemma 4.2: Let d be the degree of polynomial f. We show that the following are equivalent:

(a) For every $m, n \in \mathbb{Z}$, $m - n \mid f(m) - f(n)$.

(b) For some set S of $d+1$ consecutive integers, $m - n \mid f(m) - f(n)$ for all $m, n \in S$.

(c) There are $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ with

$$f(x) = a_n l_n \binom{x}{n} + a_{n-1} l_{n-1} \binom{x}{n-1} + \cdots + a_0 l_0 \binom{x}{0}.$$

Proof. The assertions $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$ are clear from Lemma 4.1.

Suppose (b) holds. First assume that $S = \{0, 1, \ldots, n\}$. We inductively build the sequence $a_0, a_1, \ldots$ so that the polynomial

$$P_m(x) = a_m l_m \binom{x}{m} + a_{m-1} l_{m-1} \binom{x}{m-1} + \cdots + a_0 l_0 \binom{x}{0}$$

matches the value of $f(x)$ at $x = 0, \ldots, m$. Define $a_0 = f(0)$; once $a_0, \ldots, a_m$ have been defined, let

$$a_{m+1} = \frac{f(m+1) - P_m(m+1)}{l_{m+1}}.$$

Note this is an integer since $m+1 | P_m(m+1) - P_m(0)$ by Lemma 4.1, $m+1 | f(m+1) - f(0)$ by hypothesis, and $f(0) = P_m(0)$. Noting that $\binom{x}{m+1}$ equals 1 at $x = m+1$ and 0 for $0 \le x \le m$, this gives $P_{m+1}(x) = f(x)$ for $x = 0, 1, \ldots, m+1$. Now $P_n(x)$ is a degree n polynomial that agrees with $f(x)$ at $x = 0, 1, \ldots, n$, so they must be the same polynomial.

Now if (b) holds, then by the argument above on a translated function, (c) holds for the translated function and (a) holds; in particular, (b) holds for $S = \{0, 1, \ldots, n\}$. Use the above argument to get the desired representation in (c). □

3. Let p be a prime. Show that you can place p^3 pieces on a $p^2 \times p^2$ chessboard, so that no four form a rectangle with sides parallel to sides of the board.

   Solution There are $p(p+1) > p^2$ lines in $\mathbb{F}_p^2$. Associate with each column a point of $\mathbb{F}_p^2$, associate each row with a line, and place a piece on a square only if the point associated to the column is on the line associated to the row. Two points determine a unique line so there is no rectangle.