# Team Contest Round 2

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All answers must be proved!

## 1 Algebra

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$xyf(x+y) = (x+y)f(x)f(y), \qquad |f(x)| \geq x$$
for all $x, y \in \mathbb{R}$.

2. Show that if $a, b, c$ are positive and $a + b + c = 1$, then
$$(9abc+1)\left(\frac{1}{(1-a)^2} + \frac{1}{(1-b)^2} + \frac{1}{(1-c)^2}\right) \geq 9.$$

3. Let $p$ be a prime and $n_1, \ldots, n_k$ be integers. Let
$$S = \left|\sum_{j=1}^{k} \cos\left(\frac{2\pi n_j}{p}\right)\right|.$$
Prove that either $S = 0$ or
$$S \geq k\left(\frac{1}{2k}\right)^{\frac{p-1}{2}}.$$

## 2 Combinatorics

1. There is a 9 by $5\sqrt{3}$ piece of paper with wrap around both on top/bottom and on the sides. What is the maximum number of points you can place so that no two are less than 1 away?

2. Fatty McButterpants is bored and decides to toss stones into two buckets. Both buckets start with no stones, and every second he throws a stone into the first bucket, the second bucket, or no buckets, each with probability $\frac{1}{3}$. After $n$ seconds, what is the probability that the number of stones in each bucket is divisible by 3?

3. Define a graph tiling of a graph $G$ by a graph $T$ to be a method of partitioning the edges of $G$ into sets such that the edges of each set form a graph equivalent to $T$. We say $G$ is tileable by $T$ if there exists a graph tiling of $G$ by $T$. Prove that for even $a$, $K_n$ is tileable by $K_{a,a}$ if and only if $n = 2ka^2 + 1$ for some integer $k$.

# 3 Geometry

1. Let $\omega_1$ be a circle smaller than and internally tangent to $\omega_2$ at $T$. A line $l$ is tangent to $\omega_1$ at $T'$ and hits $\omega_2$ at $A$ and $B$. If $AT' = 7$ and $BT' = 56$, find the maximum possible area of $\triangle ATB$.

2. Let $\omega_1$ and $\omega_2$ be externally tangent at $T$ and have centers $O_1$ and $O_2$. A common external tangent of these two circles intersects $\omega_1$ at $A_1$ and $\omega_2$ at $A_2$. Say $P$ is such that $TP$ is perpendicular to $O_1O_2$. Let $PA_1$ and $PA_2$ intersect $\omega_1$ and $\omega_2$ at $B_1$ and $B_2$. Prove that the circumcircle of $\triangle PB_1B_2$ is tangent to $\omega_1$ and $\omega_2$.

3. Given two rays $\vec{XP}$ and $\vec{YP}$, a triangle $ABC$ of fixed dimensions is placed with $B$ on ray $\vec{XP}$ and $C$ on ray $\vec{YP}$. Prove the locus of $A$ lies on an ellipse.

# 4 Number Theory

1. For $n \in \mathbb{N}$, define
$$p_n = \prod_{d|n} d, \qquad q_n = \prod_{1\le k\le n,\, \gcd(k,n)=1} k.$$
Prove that there exists a sequence $x_1, x_2, \ldots$ such that both the following hold:
   (a) For every $m \in \mathbb{Z}$ there exists a unique $i$ such that $x_i = m$.
   (b) For every $n \in \mathbb{N}$, $p_n^{q_n} \mid x_1 + \cdots + x_n$.

2. Suppose $f(x)$ is a polynomial of degree $d$ that takes integer values for each integer $x$, and
$$m - n \mid f(m) - f(n)$$
for all pairs of integers $(m, n)$ satisfying $0 \le m, n \le d$. Is it necessarily true that
$$m - n \mid f(m) - f(n)$$
for all pairs of integers $(m, n)$?

3. Let $p$ be a prime. Show that you can place $p^3$ pieces on a $p^2 \times p^2$ chessboard, so that no four form a rectangle with sides parallel to sides of the board.