Team Contest Round 3

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All answers must be in reduced form; all formulas must be in closed form unless specified otherwise.

1 Algebra

1. [2] a, b, c are positive numbers satisfying $abc = 48$. Find the minimum possible value of $(a+1)(b+2)(c+3)$.

2. [3] Define $m \circ n = \frac{m+n}{mn+4}$. Calculate

$$(((2010 \circ 2009) \circ 2008) \cdots \circ 1) \circ 0.$$

3. [3] Find the sum of the squares of the reciprocals of the roots of $x^5 + 3x^4 + 5x^3 + 7x^2 + 9x + 11 = 0$.

4. [4] How many ordered pairs of real polynomials $(f(x), g(x))$ are there so that

$$f(x)^2 + g(x)^2 = \frac{x^{20} - 1}{x^2 - 1}?$$

5. [4] Suppose P is polynomial of degree at most 7 so that

$$P\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right) = -\frac{\sqrt{6}-\sqrt{2}}{4}$$
$$P\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$$
$$P\left(\frac{1}{2}\right) = \frac{1}{2}$$
$$P\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = -\frac{\sqrt{2}+\sqrt{6}}{4}$$
$$P\left(-\frac{\sqrt{2}+\sqrt{6}}{4}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$P\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$
$$P\left(-\frac{1}{2}\right) = -\frac{1}{2}$$
$$P\left(-\frac{\sqrt{6}-\sqrt{2}}{4}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Find $P(5/4)$.

6. [5] Positive reals $a_1 \leq \ldots \leq a_n$, where $n \geq 4$, are such that for any 4 distinct indices i, j, k, l, the numbers a_i, a_j, a_k, a_l are sides lengths of a (possibly degenerate) quadrilateral. Find the maximum possible value of
$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$
(3 points) and determine all equality cases (2 points).

2 Combinatorics

1. [2] How many ways are there to place 5 toy cars of length 1, 2, 3, 4, and 5 inches at integer intervals on a 20 inch highway?
2. [2] How many 3-element subsets of $\{1, 2, \ldots, 20\}$ have their sum divisible by 4?
3. [3] The roads in a city form a square grid of 5 blocks. Hence, in total there are 60 block-lengths of roads. Find the longest car trip, in block-lengths, that one can take without tracing any stretch of road twice. You may go to the same intersection twice, and start and end at different locations.
4. [4] Let $x_1, \ldots, x_n$ be n points (in that order) on the circumference of a circle. A person starts at the point x_1 and walks to one of the two neighboring points with probability $\frac{1}{2}$ for each. The person continues to walk in this way, always moving from the present point to one of the two neighboring points with probability $\frac{1}{2}$ for each. Find the probability p_i (in terms of i and n) that the point x_i is the last of the n points to be visited for the first time.
5. [5] Let S be the set of all triplets of natural numbers (a, b, c) such that $a+b+c = 2010$. Calculate $\sum_{(i,j,k)\in S} ijk$. You may express your answer as $\binom{m}{n}$.
6. [5] PikaTim Chu is running a "Pokémon or Fruit" game. He has a list of $2n$ names; n of them are Pokémon and n of them are fruit. He announces the names one at a time. Suppose that you start with \$1 and you are totally incapable of distinguishing between Pokémon and fruit. Whenever a name is announced you may bet (at even odds) on whether the name is that of a Pokémon or fruit, with an amount equal to any fraction of the money you currently have. In terms of n, what is the maximum amount of money you can be guaranteed to have by the end of the game?

3 Geometry

1. [2] $ABCDEF$ is an equiangular hexagon with $AB = 1, BC = 3, CD = 3, DE = 2$. Find EF.

2. [3] A is the center of circle Ω with radius 1 and B, C are on Ω such that ΔABC is equilateral. Let ω be the circumcircle of ΔABC. Find the area of the region inside ω but outside Ω.

3. [3] Cyclic quadrilateral $ABCD$ is inscribed in circle of radius 5. $AB = 6$, $BC = 7$, $CD = 8$. Find DA.

4. [4] Find the maximum number of nonoverlapping circles of radius 5 that can fit on the surface of a cylinder with circumference 16 and height 2010. (The figures are circles when the cylinder is unrolled.)

5. [4] Let ABC be an equilateral triangle of side length 1. $\mathcal{C}$ is a smooth nonintersecting curve going from a point on AB to a point on AC such that $\mathcal{C}$ divides ABC into two figures of equal area. Find the minimal possible length of $\mathcal{C}$.

6. [5] In $\triangle ABC$, I is the incenter and D ia the point of tangency of the incircle with BC. Draw the circle with diameter AI; let Q and P be its second intersections with lines BI and CI. Knowing that $BI = 6$, $CI = 5$, $DI = 3$ find $(DP/DQ)^2$.

4 Number Theory

1. [1] Is 2011 prime?

2. [2] Find the least m such that m and $m + 1$ both have sum of digits divisible by 14.

3. [3] Find all primes p such that $2^{p+9} - 1$ is divisible by p.

4. [3] Let the 119th, 120th, and 121st digits to the right of the decimal point in $\frac{1}{9999999993}$ be x, y, z, respectively. Given that $1.3 \times 10^{10} < 7^{12} < 1.39 \times 10^{10}$, find $100x + 10y + z$.

5. [3] Find the least positive integer such that $\frac{\tau(n^2)}{\tau(n)} = 3$, where $\tau(n)$ denotes the number of positive integer divisors of n (including 1 and n).

6. [4] Evaluate
$$\sum_{i=1}^{100} \varphi(i) \cdot \left\lfloor \frac{100}{i} \right\rfloor.$$

7. [5] Suppose P is a polynomial with integer coefficients. Let N be the number of possibilities for the sequence $(P(0), P(1), \ldots, P(11^{13} - 1))$ modulo 11^{13}. Find $\log_{11} N$.

5 Grab Bag

1. [1/2] What is the exact title of the 13th chapter of Problems from the Book?
2. [1/2] What year was the artofproblemsolving website started?
3. [1/2] What is the course number for mathematics at MIT?
4. [1/2] How many posts does Altheman have on AoPS? (If you overestimate you get 0 points. The team who guesses a number less than or equal to the actual number, and is closest, gets $\frac{1}{2}$ point.)
5. [1/2] What is the minimum number of people in the same room so that there's at least a $\frac{1}{2}$ chance that 2 have the same birthday? (Assume no one in the room is born on February 29, and that the probability of a person being born on any given day is $\frac{1}{365}$.)
6. [6] The following problems are Hilbert's 1st, 3rd, 6th, 8th, 10th, and 17th problems. Label the problem with the correct number. 1 point for each correct one, -1 point for each incorrect one, and 0 for a blank. If the total point value on this problem is negative, your team gets 0 points on this problem instead.
   (a) Algorithmically solving Diophantine equations
   (b) Axiomatize physics
   (c) Continuum hypothesis
   (d) Cutting polyhedron and reassembling them into one of equal volume
   (e) Expressing nonnegative rational functions as quotients of sums of squares
   (f) Riemann hypothesis and Goldbach's conjecture