AwesomeMath Team Contest Round 2

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Question 1 (in each section) must be presented by someone with ≤ 3 points, question 2 must be presented by someone with ≤ 5 points, and the rest can be presented by anyone. (Points = sum of levels of classes.)

All problems must be proved!

Algebra

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for any real x, $f(10 + x) = f(10 - x)$ and $f(20 + x) = -f(20 - x)$. Prove that f is odd and periodic.

2. Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x + y) = f(x)f(y) - c \sin x \sin y$$

for all reals x, y where $c > 1$ is a given constant.

3. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{n - 1 + x_i} \leq 1.$$

4. Consider the space $G = (\mathbb{Z}/2012\mathbb{Z})^n$ of ordered n-tuples of integers modulo 2012, with addition and scalar multiplication defined componentwise. We say that $a \in G$ is a *combination* of $a_1, \ldots, a_m \in G$ if there exist integers $c_1, \ldots, c_m$ such that $a = c_1a_1 + \cdots + c_ma_m$. (The sum is taken modulo 2012.) Find the maximal integer N with the following property: there exists a sequence $a_1, \ldots, a_N$ of nonzero elements of G, such that for any $1 \leq k \leq N$, a_k is not a combination of $a_1, \ldots, a_{k-1}$.

Combinatorics

1. Mario and Luigi happen upon a 101×101 array of giant white mushrooms in Mushroom Kingdom. They may start on any two mushrooms, and in one step, one of them may jump to one of up to four neighboring mushrooms. When one of them jumps on a white mushroom, it turns green; when he jumps on a green mushroom it turns yellow; when he jumps on a yellow mushroom it turns red; when he jumps on a red mushroom the mushroom eats him alive. They pass the level if all mushrooms are red (and they are instantly teleported away). Prove that Mario and Luigi are doomed to be eaten alive.

2. An ant is at 0 on the number line. The probability it goes forward one space is p and probability it goes backward one space is $1 - p$. Find with proof the probability that after leaving 0, it never returns to 0.

3. Two players are playing a game on a number line. Each turn, the first player marks two unmarked integers with X, and the second player marks three unmarked integers with O. The second player wins if there are O's on k consecutive integers. Find all k for which the second player can guarantee a win.

4. The game of Chocolate proceeds as follows: Start with 2011 pieces of chocolate on the table. The first player eats 1 chocolate. On a player's turn, he or she must either eat as many chocolates as other the player did on the previous turn, or eat twice as many chocolates. The person who cannot do so loses. Assuming that neither player gets indigestion, which player has a winning strategy?

5. A class consists of 33 students. Each student is asked how many other students in the call have his first name, and how many have his last name. It turns out that each number from 0 to 10 occurs among the answers. Show that there are 2 students in the class with the same first and last name.

Geometry

1. Let ABC be a triangle with circumcenter O, and let O_a, O_b, O_c be circumcenters of triangles BOC, COA, and AOB, respectively. If triangle ABC is acute-angled, prove that the circumradius of triangle $O_aO_bO_c$ is greater or equal than the circumradius of ABC.

2. Let ABC be a triangle with orthocenter H, and let A', B', C' be the reflections of the vertices A, B, C with respect to the sidelines BC, CA, and AB, respectively. Prove that the circumcircles of triangles $AB'C'$, $BC'A'$, $CA'B'$ are concurrent.

3. Let ABC be a triangle with circumcircle Γ and let D be the tangency point of its incircle with the side BC. Furthermore, let P be a point lying on the side BC and take Q to be the second intersection of the line AP with Γ.

   (a) Prove that the circumcircle of triangle DPQ passes through a point that is independent of the position of P on BC.

   (b) Prove that the fixed point from part (a) is the tangency point of Γ with the circle Ω tangent simultanously to AB, AC and internally to Γ.

4. Let ABC be an acute triangle with circumcenter O and circumradius R. Let AO meet the circumcircle of OBC again at D, BO meet the circumcircle of OAC at E and CO meet the circumcircle of OAB at F. Show that $OD \cdot OE \cdot OF \geq 8R^3$.

Number Theory

1. For a positive integer n, let $\sigma(n)$ denote the sum of its divisors and $\sigma'(n)$ denote the sum of its even divisors. Find, with proof, all possible values of $\frac{\sigma'(n)}{\sigma(n)}$.

2. Find all pairs of positive integers (x, y) satisfying $x^2 + y^2 - 5xy + 5 = 0$.

3. Let P be a polynomial with rational coefficients and of degree at least 2. We say that x is a *periodic point* of P if

$$\underbrace{P(P(\cdots P(}_{n}x)\cdots)) = x$$

for some $n \geq 1$. Prove that there are only finitely many rational periodic points of P.

4. Let N be a natural number and $x_1, x_2, \cdots, x_n$ natural numbers such that $1 < x_i < N$ for all i and $\mathrm{lcm}(x_i, x_j) > N$ for all $i \neq j$. Show that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 2.$$

5. Find all primes p such that there exist positive integers a, b with $p = \frac{b}{4}\sqrt{\frac{2a-b}{2a+b}}$.