AwesomeMath Team Contest Round 3

AWESOME Staff

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Rule change: No point-based restrictions on who can present which problem.

Algebra

1. Define a sequence of real numbers $\{a_n\}_{n\geq 0}$ by $a_0 = 2011$ and

$$a_{n+1} = \frac{a_n + \sqrt{a_n^2 - 4}}{2}.$$

   Find with proof $\lfloor a_{1516545} \rfloor$.

2. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$ we have $f(x+1) = f(x) + 1$ and $f(x^2) = f(x)^2$.

3. Given a, b, $c \geq 1$, Prove that:

$$7(ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca) \geq (a + b + c)^3 - 27abc$$

4. Let x, y be reals such that the interval $[x, y]$ doesn't contain an integer. Show that there exists a positive integer N such that the interval $[Nx, Ny]$ also does not contain an integer, and $Nx - Ny \geq \frac{1}{6}$.

5. Let p and q be distinct odd primes. Define the polynomials f and g by

$$f(X) = pX^{p-1} + p^4X^{p-2} + \cdots + p^{(p-1)^2}X + p^{p^2}$$
$$g(X) = qX^{q-1} + q^4X^{q-2} + \cdots + q^{(q-1)^2}X + q^{q^2}$$

   Prove that the polynomial

$$h(X) = X^{pq} + q^{n^2}f(X^q) + p^{n^2}g(X^p)$$

   is irreducible over $\mathbb{Z}[X]$.

6. Find all polynomials in $\mathbb{Z}[x]$ satisfying $f(x)f(-x) = f(x^2)$.

Combinatorics

1. Let $x_1, \dots, x_n$ be n points (in that order) on the circumference of a circle. A person starts at the point x_1 and walks to one of the two neighboring points with probability $\frac{1}{2}$ for each. The person continues to walk in this way, always moving from the present point to one of the two neigboring points with probability $\frac{1}{2}$ for each. Find the probability p_i that the point x_i is the last of the n points to be visited for the first time.

2. The Judgement of the Council of Sages proceeds as follows: the king arranges the sages in a line and places either a white, black or red hat on each sage's head. Each sage can see the color of the hats of the sages in front of him, but not one of the hats of the sages behind him. Then one by one (in an order of their choosing) each sage guesses a color (every other sage is allowed to hear what one sage says). Afterwards, the king executes those sages who did not correctly guess the color of their own hat. The day before the trial, the Council meets and decides to minimize the number of executions. What is the smallest number of sages guaranteed to survive in this case?

3. Consider a 2011×2011 chessboard. Each cell is filled with one of the letters A, M, S, P. Call the resulting board "awesome" if every 2×2 subsquare contains all four letters. How many awesome boards are there? (Count all rotations and reflections as well.)

4. Let A be a set of N residues $\pmod{N^2}$. Prove that there exists a set B of N residues $\pmod{N^2}$ such that the set $A + B = \{a + b \mid a \in A, b \in B\}$ contains at least half of all residues $\pmod{N^2}$.

5. Some cities in a set T are connected with directed flights. Show that there is a set S of cities such that there are no flights between the cities in S, and for any other $c \in T$ you can reach some point in S by taking at most 2 flights.

6. A maze is an 8×8 chessboard with some adjacent cells separated by walls, such that between any two squares there is a path not passing through any wall. Given a pawn in a square of a maze and a command of the type LEFT, RIGHT, UP, DOWN it executes the command if the corresponding cell and its current cell are not blocked by a wall, and otherwise does nothing. Player A writes down a sequence of such commands (a program), and then player B constructs a maze, places a pawn into one of its cells and executes the command. Can player A make sure that at the end of the program each cell of the maze had been visited at least once by the pawn, no matter what are B's actions?

Geometry

1. Let ABC be a triangle with H its orthocenter. Define D be the point of tangency of the incircle of BCH with BC, and define E on CA and F on AB analogously. Characterize all non-right triangles ABC such that AD, BE, CF intersect at a point.

2. Let ABC be a triangle with circumcircle Γ. Let $X \in \Gamma$ such that $\angle XAB = \angle XAC$. Let Γ' be a circle tangent to AX (on the same side as C), tangent to BC (on the same side as A), and tangent internally to Γ. Show that the point of tangency of Γ' with AX is the incenter of ABC.

3. Let m, n be integers; find the maximum number of lattice points a $m \times n$ rectangle may cover.

4. Let ABC be a triangle with circumcircle Γ and let P be a point lying on the side BC. Denote by $\mathcal{T}_1$ the circle tangent to AP, BP and internally to Γ, and by $\mathcal{T}_2$ the circle tangent to AP, CP and internally to Γ. Prove that the radii of $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal if and only if P is the tangency point of the A-excircle with BC.

5. Let P be a point inside a given triangle ABC, and let A_1, B_1, C_1 be the intersections of the lines AP, BP, CP with the corresponding opposite sides. Construct three circles ω_1, ω_2, and ω_3 outside the triangle which are tangent to the sides of ABC at A_1, B_1, and C_1, respectively, and also tangent to the circumcircle of ABC. Prove that the circle tangent externally to these three circles is also tangent to the incircle of triangle ABC.

6. In a triangle ABC with circumcircle Γ, let $\mathcal{K}_a$ be the circle tangent to the sides AB, AC and to Γ internally. Denote by A' the tangency point of this circle with Γ. Similarly, define the circles $\mathcal{K}_b$, $\mathcal{K}_c$, and their tangency points with Γ, B' and C' respectively. If X, Y, Z are the intersections points of the internal angle bisectors of triangle ABC with the corresponding opposite sides of the triangle, prove that the circumcircles of triangles AXA', BYB', CZC' have a common radical axis.

Number Theory

1. Let $\{a_n\}_{n\geq 1}$ be a strictly increasing sequence of integers such that

$$a_n = 4a_{n-1} - a_{n-2}$$

for $n > 2$. If $a_4 = 194$, find with proof a_5.

2. Let $k > 1$ be an odd integer. For a positive integer n, write $f(n)$ for the greatest positive integer such that $2^{f(n)} \mid k^n - 1$. Find $f(n)$ in terms of k and n.

3. Let $a_1, a_2, \ldots a_n$ be integers with gcd=1 such that $a_i | \sum\limits_{k=1}^{n} a_k$. Prove that

$$\prod_{k=1}^{n} a_k \mid \left(\sum_{k=1}^{n} a_k \right)^{n-2} .$$

4. Let $P(x)$ be a polynomial with integer coefficients. Show that $P(n)$ is a square for every integer n if and only if $P(x) = Q(x)^2$ for another polynomial $Q(x)$ with integer coefficients.

5. Find all solutions $x, y \in \mathbb{N}$ for the equation $2^x - 5 = 11^y$.

6. For a positive integer n, let $d_k(n)$ denote the number of positive divisors of n that are congruent to k modulo 4. Evaluate

$$\lim_{N\to\infty} \frac{1}{N^{\frac{3}{2}}} \sum_{n=1}^{N} (d_1(n) - d_3(n)) \left\lfloor \sqrt{N-n} \right\rfloor .$$