Inequalities Strategies

- The sum of squares is always nonnegative. Transform into something of the form $\sum p_i \geq 0$, where each $p_i \geq 0$; for example $p_i = q_i^2$.

- Look for symmetry.

  1. If the expression is symmetric assume an ordering; assume maximal and minimal variables.
  2. Express in terms of elementary symmetric functions.
  3. If it's not symmetric, can you make it symmetric?

- Homogenize (so all terms have equal degree), or dehomogenize. Homogenizing can be useful when given homogeneous conditions such as $x + y + z = 1$.

- If the expression is linear in one variable, the extremum is attained when that variable is minimal or maximal.

- If the expression is quadratic in one variable, the extremum is attained at the vertex. Check the discriminant.

- Use algebraic or trigonometric substitutions. Look for trig identities in disguise.

  1. $xyz = 1$: Let $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$.
  2. $xyz = x + y + z + 2; x, y, z > 0$: Let $x = \frac{b+c}{a}$, $y = \frac{c+a}{b}$, $z = \frac{a+b}{c}$.
  3. $xy + yz + zx + 2xyz = 1$: Let $x = \frac{a}{b+c}$, $y = \frac{b}{c+a}$, $z = \frac{c}{a+b}$.
  4. $x^2 + y^2 + z^2 - xyz = 4$: Let $x = a + \frac{1}{a}$, $y = b + \frac{1}{b}$, $z = c + \frac{1}{c}$, with $abc = 1$.
  5. $x^2 + y^2 + z^2 + xyz = 4$: Let $x = 2\cos A$, $y = 2\cos B$, $z = 2\cos C$, where A, B, C are the angles of an acute triangle.
  6. $xy + yz + zx = 1$: Let $x = \cot A$, $y = \cot B$, $z = \cot C$, where $A + B + C = 180°$.
  7. $x + y + z = xyz$: Let $x = \tan A$, $y = \tan B$, $z = \tan C$, where $A + B + C = 180°$.
  8. a, b, c sides of a triangle: Let $a = x + y$, $b = y + z$, $c = z + a$, where $x, y, z > 0$.

- Replace variables so it becomes easier to work with. Often the lesser times each variable appears, the better!

- Look for special factorings. For example, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

- Add or subtract to numerators, for example, in Nesbitt's inequality.

- Make LHS and RHS resemble each other as closely as possible.

- Examine equality cases, and know which methods will fail.

- Induct.

- Smooth (mix) variables: move x_i together or apart; use majorization or Jensen. Keep sum or product constant and try to increase/ decrease the value of the expression. If a differentiable function has 1 flex point, then the sum of the values of f at n points with fixed sum has a maximum/minimum when $n-1$ of the values are equal.

- Test for convexity and concavity with the second derivative. A convex function attains maximum at an endpoint (boundary).

- When extrema occur for unusual parameterizations, terms may be completely dropped.

- Look for telescoping series or products.

- Reinterpret geometrically.

- Titu's Lemma with numerators made stronger or weaker.

- Isolated fudging: restrict attention to individual terms.

  1. Get each term to be a function in one variable.
  2. If $f(x) \geq g(x)$ (or vice versa) and $f(x) = g(x)$ at $x = x_0$, then $f'(x_0) = g'(x_0)$. Use this to "guess" a good function $g(x)$ to bound the given expression.
  3. Get something to add cyclically to cancel or reduce to a given quantity.
  4. Work in denominators first; use AM-GM or Cauchy.

- Asymmetric manipulation

  1. Pigeonhole principle; average principle: If one number is greater/ less than d if the average is greater/ less than d.
  2. Focus on variables greater than or less than something.
  3. Work with arbitrary weights and then choose the most appropriate at end.
  4. Casework

- Work with differences by Lagrange's identity: (for example, to deal with strong Cauchy-Schwarz-like bound.)

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1\leq i<j\leq n} (a_i b_j - a_j b_i)^2$$

- Common Cauchy-Schwarz: $\sqrt{(a+b)(a+c)} \geq a+\sqrt{bc}$, $\sqrt{ab}+\sqrt{cd} \geq \sqrt{(a+b)(c+d)}$.

- Prove the contrapositive.

- Strange relations passed off as equations can sometimes be interpreted as simple inequalities.

- Using calculus:

1. Show a function is increasing or decreasing by taking the derivative.
2. Use integral forms of inequalities.
3. Integrate both sides of a known inequality.
4. Use areas (rectangles).

- BRUTE FORCE!!! For a homogeneous inequality, clear denominators and...

  1. Look for $(a-b)^n$, Schur's inequality (strong for large powers).
  2. Look for $\sum(blah)(a-b)^m$.
  3. AM-GM/ Muirhead.