Inequalities

Rearrangement and Chebyshev Inequalities

Theorem 1 (Rearrangement). Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers (not necessarily positive) with

$$x_1 \le x_2 \le \cdots \le x_n, \text{ and } y_1 \le y_1 \le \cdots \le y_n,$$

and let σ be a permutation of $\{1, 2, \ldots, n\}$. Then the following inequality holds:

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \le x_1 y_{\sigma 1} + x_2 y_{\sigma 2} + \cdots + x_n y_{\sigma n} \le x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Proof. We prove the inequality on the right by induction on n. The statement is obvious for $n = 1$. Suppose it true for $n - 1$. Let m be an integer such that $\sigma m = n$. Since $x_n \ge x_m$ and $y_n \ge y_{\sigma n}$,

$$(x_n - x_m)(y_{\sigma n} - y_n) \ge 0 \implies$$
$$x_m y_n + x_n y_{\sigma n} \ge x_m y_{\sigma n} + x_n y_n.$$

Hence

$$x_1 y_{\sigma 1} + \cdots + x_m y_{\sigma m} + \cdots + x_n y_{\sigma n} \le x_1 y_{\sigma 1} + \cdots + x_m y_{\sigma n} + \cdots + x_n y_n.$$

By the induction hypothesis,

$$x_1 y_{\sigma 1} + \cdots + x_m y_{\sigma n} + \cdots + x_{n-1} y_{\sigma(n-1)} \le x_1 y_1 + \cdots + x_m y_m + \cdots + x_{n-1} y_{n-1}.$$

Combining these two we get the desired inequality.

To prove the LHS, apply the above with $-y_i$ instead of y_i. □

Theorem 2 (Chebyshev). Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two similarly sorted sequences. Then

$$\frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} \le \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n} \le \frac{a_1 b_1 + \cdots + a_n b_n}{n}$$

Proof. Add up the following inequalities (which hold by the Rearrangement Inequality):

$$\begin{aligned} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &\le a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ a_1 b_2 + a_2 b_3 + \cdots + a_n b_1 &\le a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ &\vdots \\ a_1 b_n + a_2 b_1 + \cdots + a_n b_{n-1} &\le a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned}$$

This gives the right-hand inequality.

By replacing y_i with $-y_i$ and using the above result we get the left-hand inequality. □

Problems

1. Powers: For $a, b, c > 0$ prove that

(a) $a^a b^b c^c \geq a^b b^c c^a$.

(b) $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$.

2. Prove the following for $x, y, z > 0$:

(a) $\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$.

(b) $\frac{xy}{z^2} + \frac{yz}{x^2} + \frac{zx}{y^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.

3. (IMO 1978/2) Let $a_1, \dots, a_n$ be pairwise distinct positive integers. Show that

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

4. (modified ISL 2006/A4) Prove that for all positive a, b, c,

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ac}{a+c} \leq \frac{3(ab+bc+ca)}{2(a+b+c)}.$$

5. (MOSP 2007) Let k be a positive integer, and let $x_1, x_2, \dots, x_n$ be positive real numbers. Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{1+x_i}\right)\left(\sum_{i=1}^{n} x_i\right) \leq \left(\sum_{i=1}^{n} \frac{x_i^{k+1}}{1+x_i}\right)\left(\sum_{i=1}^{n} \frac{1}{x_i^k}\right).$$

6. Prove that for any positive real numbers a, b, c the following inequality holds:

$$\frac{a^2+bc}{b+c} + \frac{b^2+ac}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c.$$

Convexity, Jensen's and Karamata's Inequalities

Definition 1. A function $f : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) is said to be *convex* if for any $t \in [0,1]$ and $x, y \in I$, the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

We say that $f(x)$ is *strictly convex* if equality holds only when $t = 0, 1$ or $x = y$. If the inequalities are reversed, we say that $f(x)$ is *concave* or *strictly concave*.

You may think of convexity as meaning that the line segment joining two points of the graph of f is always greater than the graph itself. Note that if you know something is convex, you do not know that it is necessarily increasing or decreasing! However, the absolute maximum of a convex function (if it exists) never occurs on the interior of the interval of definition.

We have a number of examples of convex functions:

- $f(x) = x^r$ for $r \geq 1$ and $x \geq 0$.
- $f(x) = -\log x$.

- $f(x) = 1/x$ for $x > 0$.
- $f(x) = 1/(x^2+1)$ for $x \geq 1/\sqrt{2}$.

We have other characterizations of convex functions:

Theorem 3. A function $f(x)$ is convex if and only if for any $x_1 \leq y_1 < x_2 \leq y_2$ (where $[x_1, y_2]$ is in the domain of f) we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1},$$

and strictly convex if and only if this inequality is always strict. A function $f(x)$ which is differentiable everywhere is convex if and only if $f'(x)$ is an nondecreasing function of x, and strictly convex if and only if $f'(x)$ is increasing. A function $f(x)$ which is twice differentiable everywhere is convex if and only if $f''(x) \geq 0$, and strictly convex if $f''(x) > 0$ (though not conversely in general).

It is easy to find the maximum of a convex function:

Theorem 4. If f is convex, then the maximum value of $f(x)$ on the interval $[a, b]$ is attained when $x = a$ or when $x = b$. If f is concave, then the minimum value of $f(x)$ on the interval $[a, b]$ is attained when $x = a$ or when $x = b$.

Jensen's inequality essentially extends the elementary notion of convexity to any number of variables:

Theorem 5. Let $f(x)$ be convex on an interval I, let $x_1, x_2, \ldots, x_n \in I$, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be nonnegative real numbers (weights) with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. Then

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n).$$

If $f(x)$ is strictly convex, then equality holds if and only if $\lambda_i = 1$ for an i or all the x_i are equal.

Proof. Induct on n. □

Karamata's inequality is a generalization of Jensen's.

Definition 2. The sequence $x_1 \geq x_2 \geq \cdots \geq x_n$ *majorizes* the sequence $y_1 \geq y_2 \geq \cdots \geq y_n$, denoted

$$(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n)$$

if

$$\begin{aligned} x_1 &\geq y_1 \\ x_1 + x_2 &\geq y_1 + y_2 \\ \vdots \quad &\geq \quad \vdots \\ x_1 + x_2 + \cdots + x_{n-1} &\geq y_1 + y_2 + \cdots + y_{n-1} \\ x_1 + x_2 + \cdots + x_{n-1} + x_n &= y_1 + y_2 + \cdots + y_{n-1} + y_n \end{aligned}$$

Theorem 6 (Karamata Majorization). Let $f : I \to \mathbb{R}$ be convex on I and suppose that $(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n)$, where $x_i, y_i \in I$. Then

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

The reverse inequality holds if f is concave.

Proof. Let

$$c_i = \frac{f(x_i) - f(y_i)}{x_i - y_i}, \qquad S_k = \sum_{i=1}^{k} x_i, \qquad T_i = \sum_{i=1}^{k} y_i.$$

Then

$$\begin{aligned}\sum_{i=1}^{n} f(x_i) - f(y_i) &= \sum_{i=1}^{n} c_i(x_i - y_i) \\ &= \sum_{i=1}^{n} c_i(S_i - S_{i-1} - T_i + T_{i-1}) \\ &= \underbrace{c_n(S_n - T_n)}_{0} + \sum_{i=0}^{n-1}(c_i - c_{i+1})(S_i - T_i)\end{aligned}$$

The last expression is positive since convexity implies $c_i \geq c_{i+1}$ and the majorization condition implies $S_i \geq T_i$. □

Jensen's and Karamata's inequalities are related to the idea of *smoothing*, which allows you to make moves like Jensen without necessarily the assumption of convexity. More precisely, given a desired inequality $f(x_1, x_2, \ldots, x_n) \geq 0$ say, if x_i and x_j may both be replaced by values x_i' and x_j' (often the same) such that f takes on a lesser value than before, the problem may be reduced to that case (which may for example allow an induction).

Problems

1. Trig: Let A, B, C be the angles of a triangle. Prove that
   (a) $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.
   (b) $\cos A + \cos B + \cos C \leq \frac{3}{2}$.
   (c) $\cot A + \cot B + \cot C \geq \sqrt{3}$.
   (d) $\tan A + \tan B + \tan C \geq \sqrt{3}$.
   (e) $\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$.
2. Let $0 < a, b, c < 1$. Prove that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1 - \sqrt[3]{abc}}.$$

3. Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ positive numbers whose sum is 1. Prove that

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}.$$

4. (USAMO 1977/5) If a, b, c, d, e are positive reals bounded by p and q with $0 < p \leq q$, prove that

$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$$

and determine when equality holds.

5. Let $f(x)$ be a convex function defined on an interval I, and let $x_1, x_2, x_3 \in I$. Prove that

$$\begin{aligned} & f(x_1) + f(x_2) + f(x_3) + 3f\left(\frac{x_1+x_2+x_3}{3}\right) \\ \geq\ & 2\left[f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_1}{2}\right)\right]. \end{aligned}$$

Conclude that for $a, b, c \geq 0$,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{3}{a+b+c} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c}.$$

6. (Poland) Given $a_1, \ldots, a_n \in [0,1]$, prove that

$$a_1^2 + \cdots + a_n^2 \leq \lfloor a_1 + \cdots + a_n \rfloor + \{a_1 + \cdots + a_n\}^2.$$

7. (based off Vietnam 1998) Let $x_1, x_2, \ldots, x_n$, $n \geq 2$ be positive real numbers with

$$\frac{1}{x_1+2010} + \frac{1}{x_2+2010} + \cdots + \frac{1}{x_n+2010} = \frac{1}{2010}$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 2010.$$

8. (USAMO 1998/3) Let $a_0, a_1, \ldots, a_n \in (0, \pi/2)$ be numbers such that

$$\tan(a_0 - \pi/4) + \tan(a_1 - \pi/4) + \cdots + \tan(a_n - \pi/4) \geq n - 1.$$

Prove that $\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$.

9. (Romania 1999) Show that for all positive reals $x_1, \ldots, x_n$ with $x_1 x_2 \cdots x_n = 1$, we have

$$\frac{1}{n-1+x_1} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

Cauchy-Schwarz, Hölder's, and More

Theorem 7 (Weighted Power Mean). Let $a_1, \ldots, a_n$ and $\omega_1, \ldots, \omega_n$ be positive numbers. Define the weighted power mean to be

$$\mathfrak{P}(a_1, a_2, \ldots, a_n; r) = \begin{cases} \left(\frac{\omega_1 a_1^r + \omega_2 a_2^r + \cdots + \omega_n a_n^r}{\omega_1 + \cdots + \omega_n}\right)^{1/r} & \text{if } r \in \mathbb{R}\backslash\{0\} \\ \sqrt[\omega_1 + \cdots + \omega_n]{a_1^{\omega_1} a_2^{\omega_2} \cdots a_n^{\omega_n}} & \text{if } r = 0 \\ \max(a_1, a_2, \ldots, a_n) & \text{if } r = +\infty \\ \min(a_1, a_2, \ldots, a_n) & \text{if } r = -\infty. \end{cases}$$

Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $r, s \in \mathbb{R} \cup \{\pm\infty\}$ with $r > s$. Then

$$\mathfrak{P}(a_1, a_2, \ldots, a_n; r) \geq \mathfrak{P}(a_1, a_2, \ldots, a_n; s)$$

with equality if and only if $a_1 = a_2 = \cdots a_n$.

In particular, QM-AM-GM-HM says

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Theorem 8 (Cauchy-Schwarz). Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuples of real numbers. Then

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

with equality if and only if the two n-tuples are proportional, i.e. either every $a_i = 0$ or there is a real number λ with $b_i = \lambda a_i$ for each i.

Theorem 9 (Titu's Lemma). Let $a_1, a_2, \ldots, a_n$ be real numbers and $b_1, b_2, \ldots, b_n$ positive real numbers. Then

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

Hölder's inequality is a generalization of Cauchy-Schwarz; it allows an arbitrary number of sequences of variables, as well as different weights. First we need the following:

Theorem 10 (Young). For $a, b > 0$ and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

This is a special case of the weighted AM-GM inequality.

Theorem 11 (Hölder). Let $a_1, \ldots, a_n; b_1, \ldots, b_n; \ldots; z_1, \ldots, z_n$ be sequences of nonnegative real numbers, and let $\lambda_a, \ldots, \lambda_z$ be positive reals summing to 1. Then

$$(a_1 + \cdots + a_n)^{\lambda_a}(b_1 + \cdots + b_n)^{\lambda_b} \cdots (z_1 + \cdots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z} + a_n^{\lambda_a} b_n^{\lambda_b} \cdots z_n^{\lambda_z}.$$

Proof. First we prove that

$$(a_1^p + \cdots + a_n^p)^{\frac{1}{p}} (b_1^q + \cdots + a_n^q)^{\frac{1}{q}} \geq a_1 b_1 + \cdots + a_n b_n, \tag{1}$$

when $p, q > 0$ and $\frac{1}{p} + \frac{1}{q}$, which is equivalent to the theorem statement for 2 variables.

Let $A = (a_1^p + \cdots + a_n^p)^{\frac{1}{p}}$ (also denoted $||a||_p$) and $B = (b_1^q + \cdots + b_n^q)^{\frac{1}{q}}$. Let $a'_i = \frac{a_i}{A}$ and $b'_i = \frac{b_i}{B}$. Now that we've "normalized" $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ so that $(a_1'^p + \cdots + a_n'^p)^{\frac{1}{p}} = 1$ and $(b_1'^q + \cdots + b_n'^q)^{\frac{1}{q}} = 1$, we can apply Minkowski's inequality.

$$a'_1 b'_1 + \cdots + a'_n b'_n \le \frac{1}{p}(a_1'^p + \cdots + a_n'^p)^{\frac{1}{p}} + \frac{1}{q}(b_1'^q + \cdots + b_n'^q)^{\frac{1}{q}} = \frac{1}{p} + \frac{1}{q} = 1$$

Multiplying by AB gives (1).

The general inequality follows by induction on the number of sequences. For example, passing from 2 to 3 sequences, apply Hölder to with weights $\frac{\lambda_a}{\lambda_a + \lambda_b}, \frac{\lambda_b}{\lambda_a + \lambda_b}$, then with weights $\lambda_a + \lambda_b$, λ_c. □

Theorem 12 (Minkowski). Let $p > 1$ and let $a_1, \ldots, a_n, b_1, \ldots, b_n, \ldots, z_1, \ldots, z_n$ be positive numbers. Then

$$\begin{aligned} &(|a_1|^p + \cdots + |a_n|^p)^{\frac{1}{p}} + (|b_1|^p + \cdots + |b_n|^p)^{\frac{1}{p}} + \cdots + (|z_1|^p + \cdots + |z_n|^p)^{\frac{1}{p}} \\ &\quad \ge [(|a_1 + \cdots + z_1|)^p + (|a_2 + \cdots + z_2|)^p + \cdots + (|a_n + \cdots + z_n|)^p]^{\frac{1}{p}} \end{aligned}$$

Proof. We first prove Minkowski for 2 sequences. We have

$$\begin{aligned} \sum_{k=1}^n |a_k + b_k|^p &= \sum_{k=1}^n (|a_k| + |b_k|)|a_k + b_k|^{p-1} \\ &= \sum_{k=1}^n |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^n |b_k||a_k + b_k|^{p-1} \\ &\le \left(\left(\sum_{k=1}^n |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |b_k|^p \right)^{\frac{1}{p}} \right) \left(\sum_{k=1}^n |a_k + b_k|^{(p-1)\left(\frac{p}{p-1}\right)} \right)^{\frac{p-1}{p}} \end{aligned}$$

$$\left(\sum_{k=1}^n |a_k + b_k|^p \right)^{\frac{1}{p}} \le \left(\sum_{k=1}^n |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |b_k|^p \right)^{\frac{1}{p}}$$

The general case follows by induction. □

As a corollary, $||x||_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$ is a valid norm in n-dimensional space for $p \ge 1$.

Theorem 13 (Schur). Let $a, b, c \ge 0$ and $r > 0$. Then

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \ge 0$$

with equality iff $a = b = c$ or some two are equal and the other is 0.

Proof. Suppose WLOG $a \ge b \ge c$. Rewrite the inequality as

$$(a-b)(a^r(a-c) - b^r(b-c)) + c^r(a-c)(b-c).$$

Both terms are positive. □

Theorem 14 (Muirhead). Suppose $(a_1, \ldots, a_n) \succ (b_1, \ldots, b_n)$. Then for any positive reals $x_1, \ldots, x_n$,

$$\sum_{\text{sym}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{\text{sym}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

where the sum is taken over all permutations of n variables.

Problems

1. (Nesbitt) Let $a, b, c > 0$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

2. Prove Aczel's Inequality: If $a_1^2 \geq a_2^2 + \cdots + a_n^2$ then

$$(a_1b_1 - a_2b_2 - \cdots - a_nb_n)^2 \geq (a_1^2 - a_2^2 - \cdots - a_n^2)(b_1^2 - b_2^2 - \cdots - b_n^2).$$

   Hint: Consider the determinant of the quadratic

$$(a_1x - b_1)^2 - (a_2x - b_2)^2 - \cdots - (ax_n - b_n)^2.$$

3. Prove that for all reals a, b, c,

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \leq (a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2).$$

4. The numbers $-1 \leq x_1, x_2, \ldots, x_n \leq 1$ satisfy $x_1^3 + x_2^3 + \cdots + x_n^3 = 0$. Prove that

$$x_1 + x_2 + \cdots + x_n \leq \frac{n}{3}.$$

5. Let x, y, z be nonnegative real numbers. Prove that

$$(x^2 + 1)(y^2 + 1)(z^2 + 1) \geq (x + y + z - xyz)^2.$$

6. Let a, b, c be positive reals such that $a + b + c = 1$. Prove that

$$\sqrt{ab + c} + \sqrt{bc + a} + \sqrt{ca + b} \geq 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

7. (TST 2000/1) Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers such that $a_{n+1} \geq a_n^2 + \frac{1}{5}$ for all $n \geq 0$. Prove that $\sqrt{a_{n+5}} \geq a_{n-5}^2$ for all $n \geq 5$.

8. (IMO 2004/4) Let $n \geq 3$ be an integer. Let $t_1, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

   Prove that for all i, j, k, the numbers t_i, t_j, t_k are sides of a triangle.

9. (USAMO 2009/4) For $n \geq 2$ let $a_1, a_2, \ldots, a_n$ be positive real numbers such that
$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2.$$
Prove that $\max(a_1, a_2, \ldots, a_n) \leq 4 \min(a_1, a_2, \ldots, a_n)$.

10. (TST 2010/2) Let a, b, c be positive reals such that $abc = 1$. Show that
$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

11. (Crux Mathematicorum) a, b, c, d, e are positive reals multiplying to 1. Prove that
$$\frac{a+abc}{1+ab+abcd} + \frac{b+bcd}{1+bc+bcde} + \frac{c+cde}{1+cd+cdea} + \frac{d+dea}{1+de+deab} + \frac{e+eab}{1+ea+eabc} \geq \frac{10}{3}$$

12. (IMO 2000/2) Positive reals a, b, c have product 1. Prove that
$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

13. (IMO 2003/5) Let n be a positive integer and let $x_1 \leq \cdots \leq x_n$ be real numbers. Prove that
$$\left(\sum_{1 \leq i,j \leq n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{1 \leq i,j \leq n} (x_i - x_j)^2.$$
Show that equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

14. (IMO 2008/2) Prove that if x, y, z are three real numbers, all different from 1, such that $xyz = 1$, then
$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1.$$

15. (USAMO 2004/5) Let a, b, c be positive reals. Prove that
$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

16. (USAMO 1997/5) Prove that for all positive reals a, b, c,
$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}.$$

17. (MOSP 2007) Let a, b, and c be a nonnegative real numbers with
$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2.$$
Prove that
$$ab + bc + ba \leq \frac{3}{2}.$$

18. (USAMO 2003/5) Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2b^2+(c+a)^2}\leq 8.$$

19. (ISL 2004/A5) Let $a, b, c > 0$ and $ab + bc + ca = 1$. Prove the inequality

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a}\leq\frac{1}{abc}.$$

20. (ISL 2006/A5) Let a, b, c be the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}\geq 3.$$

21. (TST 2007/3) Show that for reals x, y, z which are not all positive,

$$\frac{16}{9}(x^2-x+1)(y^2-y+1)(z^2-z+1)\geq(xyz)^3-xyz+1.$$

22. (IMO 2005/3) Prove that for all positive a, b, c with product at least 1,

$$\frac{a^5-a^2}{a^5+b^2+c^2}+\frac{b^5-b^2}{b^5+c^2+a^2}+\frac{c^5-c^2}{c^5+a^2+b^2}\geq 0.$$

23. (IMO 2006/3) Determine the least real number M such that for all reals a, b, c,

$$|a^3b+b^3c+c^3a-a^3c-b^3a-c^3b|\leq M\cdot(a^2+b^2+c^2)^2.$$

24. Consider any sequence $a_1, a_2, \ldots$ of real numbers. Show that

$$\sum_{n=1}^{\infty}a_n\leq\frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\left(\frac{r_n}{n}\right)^{1/2},$$

where $r_n=\sum_{k=n}^{\infty}a_k^2$.

25. (ISL 2004/A7) Let $a_1, \ldots, a_n$ be positive real numbers, $n > 1$. Denote by g_n their geometric mean, and by $A_1, \ldots A_n$ the sequence of arithmetic means defined by $A_k = \frac{a_1+\cdots+a_k}{k}$, $k = 1, 2, \ldots, n$. Let G_n be the geometric mean of $A_1, \ldots, A_n$. Prove that

$$n\sqrt[n]{G_n}A_n+\frac{g_n}{G_n}\leq n+1$$

and establish the cases of equality.

Reference: *Olympiad Inequalities* by Thomas Mildorf