Exam 2 Solutions

Problem 1 (ISL 1990) Prove that for any positive real numbers a, b, c, d satisfying $ab + bc + cd + da = 1$ the following inequality is true:

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Solution We rewrite the expression and apply Titu's Lemma:

$$\begin{aligned}\sum_{\text{cyc}} \frac{a^4}{a(b+c+d)} &\geq \frac{(a^2+b^2+c^2+d^2)^2}{2(ab+ac+ad+bc+bd+cd)} \\ &\geq \frac{(a^2+b^2+c^2+d^2)^2}{(a+b+c+d)^2-(a^2+b^2+c^2+d^2)} \qquad (1)\end{aligned}$$

By the Quadratic Mean-Arithmetic Mean Inequality,

$$\frac{a+b+c+d}{4} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}},$$

or

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2). \qquad (2)$$

(Alternatively, use Cauchy-Schwarz inequality.) Note that

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 \geq 0,$$

giving the inequality

$$a^2+b^2+c^2+d^2 \geq ab+bc+cd+da. \qquad (3)$$

(Alternatively, use the AM-GM inequality or Rearrangement Inequality.) Using (2) and (3), we get

$$\begin{aligned}\frac{(a^2+b^2+c^2+d^2)^2}{(a+b+c+d)^2-(a^2+b^2+c^2+d^2)} &\geq \frac{(a^2+b^2+c^2+d^2)^2}{4(a^2+b^2+c^2+d^2)-(a^2+b^2+c^2+d^2)} \\ &\geq \frac{a^2+b^2+c^2+d^2}{3} \\ &\geq \frac{ab+bc+cd+da}{3} = \frac{1}{3}\end{aligned}$$

as desired.

Problem 2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy + f(x)) = xf(y) + f(x)$$

for all $x, y \in \mathbb{R}$.

Solution Let f be a function satisfying the given conditions. Suppose that $f(x) = f(y)$. Then

$$\begin{aligned}(x+1)f(x) &= xf(y) + f(x) \\ &= f(xy + f(x)) \\ &= f(xy + f(y)) \\ &= yf(x) + f(y) \\ &= (y+1)f(x) \implies \\ (x-y)f(x) &= 0\end{aligned}$$

(Putting $f(x) = f(y)$ is useful because it makes the left-hand side of the functional equation expressible in two ways, giving us two different right-hand sides which we can equate.) From this we get that $f(x) = 0$ or $x = y$.

Plugging in $x = 0$ into the functional equation we get

$$f(f(0)) = f(0).$$

From our above obsevation, we must have $f(0) = 0$. Plugging in $y = 0$ we get

$$f(f(x)) = xf(0) + f(x) = f(x),$$

and hence

$$\text{for all } x,\ f(x) = 0 \text{ or } f(x) = x. \tag{4}$$

Suppose by way of contradiction that $f(x) = x$ and $f(y) = 0$ for some nonzero x, y. Then

$$f(xy + x) = f(xy + f(x)) = xf(y) + f(x) = x.$$

By (4), since $x \neq 0$, $xy + x = x$ and $y = 0$, contradiction. Hence the two solutions are

$$f(x) \equiv 0, f(x) \equiv x,$$

and it is easy to check that they work.

Problem 3 Let $\omega_1, \ldots, \omega_n$ be roots of unity. Suppose that

$$\frac{\omega_1 + \cdots + \omega_n}{n}$$

is the zero of some monic polynomial with integer coefficients. Prove that either $\omega_1 + \cdots + \omega_n = 0$ or $\omega_1 = \omega_2 = \cdots = \omega_n$.

Solution Suppose $\alpha := \omega_1 + \cdots + \omega_n \neq 0$ and the minimal polynomial of α over $\mathbb{Q}$ is $x^m + a_{m-1}x^{m-1} + \cdots + a_0$. The conjugates of α are in the form $\frac{\omega_1' + \cdots + \omega_n'}{n}$ where ω_i' is a conjugate of ω_i. Note the ω_i' are all roots of unity, so by the Triangle Inequality,

$$\left|\frac{\omega_1' + \cdots + \omega_n'}{n}\right| \leq \frac{|\omega_1'| + \cdots + |\omega_n'|}{n} = 1.$$

Each conjugate has absolute value at most 1, so the product p of the conjugates has absolute value at most 1. None of the conjugates are equal to 0, and $p = \pm a_0$ is an integer, so $|p| = 1$. In order for equality to hold, we must have

$$\left|\frac{\omega_1 + \cdots + \omega_n}{n}\right| = 1.$$

By the Triangle Inequality, this happens only if $\omega_1, \ldots, \omega_n$ have the same argument (angle in the complex plane). Hence $\omega_1 = \cdots = \omega_n$.