Take-home Exam

Feel free to use as much time as you'd like, and refer to your notes. If you would like your exam graded, please either type it up (preferably in LaTeX), or write it NEATLY and scan it, and send it to me at holden1@mit.edu. You may email me if you would like hints or clarifications.

Note that these questions are more difficult than the in-class tests, and may require significantly more time. If you get stuck, try reviewing class materials and doing similar problems on the problem sets before asking for hints.

Each question is worth 7 points. I will use Olympiad-style grading!

1. Prove that for any polynomial P there exists a positive integer n so that
$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k P(k) = 0.$$

2. Prove that
$$\frac{a + \sqrt{ab} + \sqrt[3]{abc}}{3} \leq \sqrt[3]{a \cdot \frac{a+b}{2} \cdot \frac{a+b+c}{3}}$$
for any $a, b, c \geq 0$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$f(f(x) + yz) = x + f(y)f(z)$$
for all x, y, z in $\mathbb{R}$.

4. Prove that the polynomial
$$f(x) = \frac{x^n + x^m - 2}{x^{\gcd(m,n)} - 1}$$
is irreducible over $\mathbb{Q}$ for all integers $n > m > 0$.

5. (Bonus, uses linear algebra) An *algebraic curve* in $\mathbb{R}^2$ is the locus of zeros of a polynomial $f(x, y)$ in two variables with degree at least 1 in each variable. A *polynomial path* in $\mathbb{R}^2$ is a parameterized path $x = x(t)$, $y = y(t)$ where $x(t)$ and $y(t)$ are polynomials in t. Prove that every polynomial path lies in some algebraic curve.