Functional Equations

Strategies

- Substitution
  1. $x = 0, 1, y, -y, \frac{1}{y}$, etc.
  2. Recursive substitutions ($y = f(x)$).
- Surjectivity/injectivity
- Monotonicity
- Algebraic properties of the domain/range.
- Casework (e.g. either $f(0) = 0$ or otherwise...)
- Replacing the desired function with something perhaps easier to use, and then working backwards later.
- Finding a solution or family of solutions, and reverse-engineering an argument. Try a proof by contradiction.
- For functions $\mathbb{N} \to \mathbb{N}$, look in other bases.
- Use induction (for example, to find the values of the function on $\mathbb{Z}$). Keep in mind how many degrees of freedom there are (for example, maybe $f(0), f(1), f(2)$ determine f for all of $\mathbb{Z}$.)
- Iterate (repeatedly apply f). Useful when the condition involves $f \circ f$. Possibly get a recursive relation.
- Approximate with a linear function: $f(x) = \lfloor cx \rfloor$.
- Cauchy's equation and variants. If $f(x+y) = f(x)+f(y)$ and f is either continuous, monotonic, or bounded on an interval, then $f(x) = cx$ for some constant c.
- Look for fixed points $f(x) = x$. What happens if there's a fixed point?
- Strategies for polynomials
  1. Roots
  2. Match degree and coefficients
  3. Use estimation; define a convergent sequence of integers.
- Symmetrize. If the LHS of a functional equation is symmetric, and the RHS is not, then the RHS equals the RHS with the variables permuted. Additional variables can help.
- Use δ-ϵ definition of continuity.

Problems

1. Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Find all functions $\mathbb{R}^* \to \mathbb{R}^*$ such that
$$f(x^2 + y) = f(f(x)) + \frac{f(xy)}{f(x)}.$$

2. Find all functions $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$ such that
$$f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x(1-x)}.$$

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$,
$$f(x - y + f(y)) = f(x) + f(y).$$

4. (ISL 2002/A2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x) + y) = 2x + f(f(y) - x)$$
for all real x and y.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f((x + y)^2) = (x + y)(f(x) + f(y)).$$

6. (ISL 1996) Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for each $m, n \in \mathbb{N}$,
$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

7. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying
$$f(x) + f(y) \le \frac{f(x + y)}{2} \text{ and } \frac{f(x)}{x} + \frac{f(y)}{y} \ge \frac{f(x + y)}{x + y}$$
for all $x, y > 0$.

8. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x + y) + f(x)f(y) = f(xy) + 2xy + 1.$$

9. Let $\mathbb{R}^+$ denote the set of positive real numbers and let $k \in \mathbb{R}^+$ be a constant. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x)f(y) = kf(x + yf(x))$$
for all positive real numbers x and y.

10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x + f(y)) - 1) = f(x) + f(x + y) - x.$$

11. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y)) = (1 - y)f(xy) + x^2y^2f(y)$$

for all real numbers x and y.

12. (IMO 2008/4) Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$\frac{f(p)^2 + f(q)^2}{f(r^2) + f(s^2)} = \frac{p^2 + q^2}{r^2 + s^2}$$

for all $p, q, r, s > 0$ with $pq = rs$.

13. Consider those functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m + n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2009)$.

14. Find all monic polynomials $f(x)$ of degree d such that $f(x)$ is a perfect dth power for every integer x.

15. (IMO 1983/1) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

    (a) $f(xf(y)) = yf(x)$ for all $x, y \in \mathbb{R}^+$.

    (b) $\lim_{x\to\infty} f(x) = 0$.

16. (IMO 1993/2) Does there exist a function f from the positive integers to the positive integers such that $f(1) = 2$, $f(f(n)) = f(n) + n$ for all n, and $f(n) < f(n + 1)$ for all n?