Functional Equations Solutions

Solutions to Problems

1. Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Find all functions $\mathbb{R}^* \to \mathbb{R}^*$ such that
$$f(x^2+y) = f(f(x)) + \frac{f(xy)}{f(x)}.$$

   Solution. Suppose $f(x) = x^2$ for all x. Then
$$(x^2+y)^2 = x^4 + \frac{x^2y^2}{x^2},$$
   or $2x^2y = 0$, an abject contradiction. We conclude that there is some x_0 with $f(x_0) \neq x_0^2$. Letting $y = f(x_0) - x_0^2$, we obtain
$$f(f(x_0)) = f(f(x_0)) + \frac{f(x_0y)}{f(x_0)},$$
   so that the fraction would be zero, contradicting the given range.

2. Find all functions $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$ such that
$$f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x(1-x)}.$$

   Solution. We employ the facts that
$$\frac{1}{1-\frac{1}{1-x}} = 1 - \frac{1}{x} \text{ and } \frac{1}{1-\left(1-\frac{1}{x}\right)} = x$$
   to obtain
$$f\left(\frac{1}{1-x}\right) + f\left(1-\frac{1}{x}\right) = 3 - x - \frac{1}{x} \text{ and } f\left(1-\frac{1}{x}\right) + f(x) = 2 + x - \frac{1}{1-x},$$
   which gives us three equations in three unknowns. Subtracting the second equation from the third, we obtain
$$f(x) - f\left(\frac{1}{1-x}\right) = 2x - 1 + \frac{1}{x} - \frac{1}{1-x}.$$
   Finally, adding this to the first and dividing by 2 we get
$$f(x) = x + \frac{1}{x},$$
   the only solution.

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$,

$$f(x - y + f(y)) = f(x) + f(y).$$

Solution. We will define $g(x) = f(x) - x$, so that our equation becomes

$$g(x + g(y)) = g(x) + y.$$

Fixing x and letting y run through the integers we see that g is surjective. Let t be such that $g(t) = 0$; then

$$g(x) = g(x + g(t)) = g(x) + t,$$

so $t = 0$ and $g(0) = 0$. We may also find k with $g(k) = 1$. Setting y to be this k we obtain

$$g(x + 1) = g(x) + k$$

for any x, so that $g(x) = kx$ for all x. Surjectivity forces $k = \pm 1$, and we immediately see that $g(x) = \pm x$ (or $f(x) = 0$ or $2x$) are the only solutions.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all real x and y.

Solution. Setting $y = -f(x)$ we obtain

$$f(0) = 2x + f(f(-f(x)) - x), \text{ i.e. } f(f(-f(-x)) - x) = f(0) - 2x,$$

so that as we run x through all the reals we obtain surjectivity for $f(x)$. Let r be such that $f(r) = 0$; then

$$f(y) = f(f(r) + y) = 2r + f(f(y) - r), \text{ or } f(f(y) - r) = (f(y) - r) - r.$$

However $f(y) - r$ runs through all the reals, so for any x we may say $f(x) = x - r$, and all such solutions work in the original equation.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f((x + y)^2) = (x + y)(f(x) + f(y)).$$

Solution. Setting $y = 0$ we obtain $f(x^2) = xf(x)$, so $f(0) = 0$ and also

$$(x + y)(f(x) + f(y)) = f((x + y)^2) = (x + y)f(x + y).$$

If $x + y \neq 0$ this immediately gives Cauchy's equation $f(x) + f(y) = f(x + y)$. If $x + y = 0$ with $x \neq 0$, then

$$xf(x) = f(x^2) = (-x)f(-x)$$

so that

$$f(x) + f(-x) = 0 = f(0);$$

the case $x = y = 0$ is clear. We conclude that for any x, y,

$$\begin{aligned}(x+y)(f(x)+f(y)) &= f((x+y)^2) = f(x^2+2xy+y^2) \\ &= f(x^2)+f(2xy)+f(y^2) = xf(x)+f(2xy)+yf(y),\end{aligned}$$

so that

$$f(2xy) = xf(y) + yf(x).$$

Setting $y = 1$ we obtain

$$2f(x) = f(2x) = xf(1) + f(x),$$

so $f(x) = kx$ where $k = f(1)$, and these are all solutions.

6. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers. Determine whether or not there exists a bijective function $f : \mathbb{N} \to \mathbb{N}$ such that for each $m, n \in \mathbb{N}$,

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

Solution. There exist infinitely (even uncountably) many such.

Let us rewrite the given equation as

$$f\left(\frac{(3m+1)(3n+1)-1}{3}\right) = \frac{(4f(m)+1)(4f(n)+1)-1}{4}.$$

Then what we seek is a bijection $g : S \to T$ where S is the set of $3k+1$ integers and T is the set of $4k+1$ integers, such that

$$g(xy) = g(x)g(y).$$

Let $\Pi_{a,m}$ denote the set of primes congruent to a mod m. It is an elementary fact that $\Pi_{1,3}, \Pi_{2,3}, \Pi_{1,4}$, and $\Pi_{3,4}$ are all infinite, so we may choose bijections $\Pi_{1,3} \to \Pi_{1,4}$ and $\Pi_{2,3} \to \Pi_{3,4}$. From these we may define a function $g : S \to T$ mapping prime factorizations to their image under the two bijections. This works because an integer is of the form $3k+1$ if and only if it is factored into some number of $3k+1$ primes and an even number of $3k+2$ primes, and similarly for an integer of the form $4k+1$. Therefore we have obtained our solution (and the uncountably many bijections between the sets of primes yield the claimed uncountably many solutions).

7. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$f(x) + f(y) \le \frac{f(x+y)}{2} \text{ and } \frac{f(x)}{x} + \frac{f(y)}{y} \ge \frac{f(x+y)}{x+y}$$

for all $x, y > 0$.

Solution. To avoid confusion we set $g(x) = -f(x)/x$, and look for solutions to

$$xg(x) + yg(y) \ge \left(\frac{x+y}{2}\right) g(x+y) \text{ and } g(x) + g(y) \le g(x+y).$$

Substituting $y = x$ we obtain

$$2xg(x) \geq xg(2x) \text{ and } 2g(x) \leq g(2x),$$

so that $g(2x) = 2g(x)$. From this we obtain $g(2^n x) = 2^n g(x)$ for any x. What's more,

$$g(nx) \geq g(x) + g((n-1)x) \geq 2g(x) + g((n-2)x) \geq \cdots \geq ng(x)$$

for any positive integer n. Additionally,

$$5xg(x) = xg(x) + 2xg(2x) \geq \frac{3}{2}xg(3x) \geq \frac{9}{2}xg(x),$$

so $10g(x) \geq 9g(x)$ and $g(x) \geq 0$ for all x. Finally, whenever $y > x$ we have $g(x) + g(y-x) \leq g(y)$, and so $g(x)$ is nondecreasing.

Let x be a positive real. Suppose we have positive integers n, n', m, m' with

$$\frac{2^{n'}}{m'} \geq x \geq \frac{m}{2^n}.$$

Then

$$\frac{2^{n'}}{m'}g(1) = \frac{1}{m'}g\left(2^{n'}\right) \geq g\left(\frac{2^{n'}}{m'}\right) \geq g(x) \geq g\left(\frac{m}{2^n}\right) \geq mg\left(\frac{1}{2^n}\right) = \frac{m}{2^n}g(1).$$

Since these fractions may be chosen arbitrarily close to one another, we conclude that $g(x) = g(1)x$. The given inequalities work as well; the first is Titu's Lemma and the second is an equality. Thus $f(x) = -ax^2$ for $a \geq 0$ are the only solutions.

8. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x)f(y) = f(xy) + 2xy + 1.$$

Solution. Setting $x = y = 0$, we obtain $f(0)^2 = 1$, of $f(0) = \pm 1$. However, if $f(0) = 1$ then we may take $y = 0$ and obtain $2f(x) = f(x) + 1$, or $f(x) = 1$ for all x, a contradiction. Therefore we conclude that $f(0) = -1$.

Next, we look at $(x, y) = (-1, 1)$, obtaining

$$-1 + f(1)f(-1) = f(-1) - 1,$$

or

$$f(-1)(f(1) - 1) = 0.$$

Case (a). If $f(1) = 1$ then using $(x, y) = (x-1, 1)$ we obtain

$$f(x) + f(x-1) = f(x-1) + 2x - 1,$$

or $f(x) = 2x - 1$, our first solution.

Case (b). If $f(-1) = 0$, we may take $(x, y) = (-1, -1)$ and get $f(-2) = f(1) + 3$, and then with $(x, y) = (-2, 1)$ we get

$$f(-2)f(1) = f(-2) - 3.$$

Combining these, we obtain

$$f(-2)(f(1) - 1) = 0.$$

Case (bi). Suppose $f(1) = 0$. We obtain

$$f(x) = f((x-1)+1) = f(x-1) + 2x - 1 = f(-x) - 2x + 1 + 2x - 1 = f(-x),$$

so that $f(x)$ is an even function. Finally, taking $(x, y) = (x, \pm x)$, we get

$$-1 + f(x)^2 = f(x - x) + f(x)f(-x) = f(-x^2) - 2x^2 + 1$$

and

$$f(2x) + f(x)^2 = f(x^2) + 2x^2 + 1 = f(-x^2) + 2x^2 + 1,$$

so we get $f(2x) = 4x^2 - 1$, $f(x) = x^2 - 1$, a solution.

Case (bii). Suppose $f(1) = -2$. In this case we get

$$f(x+1) - 2f(x) = f(x) + 2x + 1, \text{ or } f(x) = 3f(x-1) + 2x - 1.$$

As $f(-1) = 0$, we have

$$f(x) = 3(f(-x) - 2x + 1) + 2x - 1 = 3f(-x) - 4x + 2.$$

Swapping the roles of x and $-x$ here, we get $f(-x) = 3f(x)+4x+2$, a system of two equations in two unknowns that we may use to solve for $f(x)$, getting $f(x) = -x-1$, the third solution.

9. Let $\mathbb{R}^+$ denote the set of positive real numbers and let $k \in \mathbb{R}^+$ be a constant. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)f(y) = kf(x + yf(x))$$

for all positive real numbers x and y.

Solution. Suppose that $f(x)$ is injective. Setting $x = a$ and $y = 1$, we obtain

$$f(a)f(1) = kf(a + f(a)).$$

Setting $x = 1$ and $y = a$, we get

$$f(1)f(a) = kf(1 + af(1)).$$

We conclude by injectivity that $a + f(a) = 1 + af(1)$, or

$$f(a) = 1 - a(f(1) - a),$$

or in other words $f(x) = 1 + cx$ for some constant c. In order for this to be a solution, we need

$$(1 + cx)(1 + cy) = k(1 + c(x + y(1 + cx))) = k(1 + cx)(1 + cy).$$

Therefore this is a solution if and only if $k = 1$.

Now suppose that $f(x)$ is not injective, so $f(a) = f(b) = c$ for some $a < b \in \mathbb{R}^+$. We claim that $f(x) = c$ for all positive x.

First, for all y we have

$$f(a)f(y) = kf(a + yf(a)) = kf(a + cy)$$

and

$$f(b)f(y) = kf(b + yf(b)) = kf(b + cy),$$

so

$$f((a - b) + b + cy) = f(a + cy) = f(b + cy).$$

We conclude that $f(x)$ is *periodic* of period $a - b$ for all $y \geq b$.

Now suppose we have x_1 and x_2 with $f(x_1) > f(x_2)$. We conclude that for any y, $f(x_1)f(y) \neq f(x_2)f(y)$. However we may choose y so large that both $x_1 + yf(x_1)$ and $x_2 + yf(x_2)$ are greater than or equal to b, and also that

$$[x_1 + yf(x_1)] - [x_2 + yf(x_2)] = (x_1 - x_2) + y(f(x_1) - f(x_2)) = n(a - b)$$

for some positive integer n, so that in fact

$$f(x_1)f(y) = f(x_1 + yf(x_1)) = f(x_2 + yf(x_2)) = f(x_2)f(y),$$

a contradiction so that in fact $f(x)$ is a constant. Finally, if $f(x) = c$ then $c^2 = kc$ shows that $f(x) = k$. These families comprise all solutions.

10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x + f(y)) - 1) = f(x) + f(x + y) - x.$$

Solution. Let us first prove that $f(x)$ is injective. If $f(y) = f(y')$ for some $y \neq y'$, we have

$$\begin{aligned} f(x) + f(x + y) - x &= f(f(x + f(y)) - 1) \\ &= f(f(x + f(y')) - 1) \\ &= f(x) + f(x + y') - x, \end{aligned}$$

so that $f(x + y) = f(x + y')$ and f is periodic with period $y - y' = p$. However,

$$\begin{aligned} f(x) + f(x + y) - x &= f(f(x + f(y)) - 1) \\ &= f(f(x + p + f(y)) - 1) \\ &= f(x + p) + f(x + p + y) - x - p \\ &= f(x) + f(x + y) - x - p, \end{aligned}$$

so $p = 0$, a contradiction so that f is injective.

Now let y be arbitrary and set $x = f(y) - f(0)$. We obtain

$$\begin{aligned} f(x) + f(x+0) - x &= f(f(x + f(0)) - 1) \\ &= f(f(0 + f(y)) - 1) \\ &= f(0) + f(0+y) - 0 \\ &= f(0) + f(0) + x, \end{aligned}$$

so that

$$f(y) = f(0) + x = f(x).$$

By injectivity, $x = y$, and so $f(y) = y + f(0) = y + c$. Now we plug back into our original equation with $x = y = 0$:

$$2c = f(f(c) - 1) = f(2c - 1) = 3c - 1,$$

so $c = 1$ and $f(x) = x + 1$ is the only solution.

11. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y)) = (1 - y)f(xy) + x^2y^2f(y)$$

for all real numbers x and y.

Solution. We observe that $f(x) = 0$ is a solution, and assume that $f(x)$ is not identically 0 from now on. Setting $x = 1$ we get

$$f(f(y)) = (1 - y)f(y) + y^2f(y) = (1 - y + y^2)f(y),$$

and setting $y = 1$, we obtain $f(xf(1)) = x^2f(1)$. If $f(1) \neq 0$, we are forced to have $f(x) = x^2/f(1)$ for all x, so $f(1) = 1/f(1)$ and $f(x) = \pm x^2$. This directly contradicts our first equation:

$$\pm y^4 = (1 - y + y^2)(\pm y^2).$$

We conclude that $f(1) = 0$. Setting $y = 1$ in the first equation, we get

$$f(0) = f(1) = 0,$$

so $f(0) = 0$.

We claim these are the only two values of 0. Indeed, if $f(y) = 0$, then we have

$$0 = f(xf(y)) = (1 - y)f(xy) + x^2y^2f(y) = (1 - y)f(xy),$$

so that either $y = 0, 1$ or $f(x)$ is identically 0, contradicting our assumptions.

Next we use $y = 1/x$.

$$f(xf(1/x)) = (1 - 1/x)f(1) + f(1/x) = f(1/x),$$

so for any $x \neq 0$ we obtain $f(f(x)/x) = f(x)$.

Suppose now we have two values $f(x) = f(y) \neq 0$. We argue

$$(1 - x + x^2)f(x) = f(f(x)) = f(f(y)) = (1 - y + y^2)f(y) = (1 - y + y^2)f(x).$$

So $y = x, 1 - x$. However then we have

$$\frac{f(x)}{x} = x, x - 1,$$

so for each x either $f(x) = x^2$ or $f(x) = x - x^2$. The latter case is a solution if it holds for all x.

Suppose now that some $f(x) = x^2$. We know

$$f(x^2) = f(f(x)) = (1 - x + x^2)f(x) = x^2 - x^3 + x^4.$$

If $f(x^2) = x^4$, then $x^2 - x^3 = 0$ in which case $x = 0, 1$. If $f(x^2) = x^2 - x^4$, then $2x^4 - x^3 = 0$, so $x = 0, \frac{1}{2}$. We already know that $f(1) = 0$, and moreover $0^2 = 0 - 0^2$ and $(\frac{1}{2})^2 = \frac{1}{2} - (\frac{1}{2})^2$, so actually we may conclude that $f(x) = x - x^2$ in any case. Thus (in the nonzero case) this is the only solution.

12. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$\frac{f(p)^2 + f(q)^2}{f(r^2) + f(s^2)} = \frac{p^2 + q^2}{r^2 + s^2}$$

for all $p, q, r, s > 0$ with $pq = rs$.

Solution. Setting $p = q = r = s = 1$ we obtain $f(1)^2 = f(1)$ and so $f(1) = 1$. Now let $x > 0$ and $p = x$, $q = 1$, $r = s = \sqrt{x}$ to obtain

$$\frac{f(x)^2 + 1}{2f(x)} = \frac{x^2 + 1}{2x}.$$

This rearranges into

$$xf(x)^2 + x = x^2 f(x) + f(x),$$

or

$$(xf(x) - 1)(f(x) - x) = 0.$$

Therefore either $f(x) = x$ or $f(x) = 1/x$ for every $x > 0$.

The functions $f(x) = x$ and $f(x) = 1/x$ both satisfy the conditions of the problem; we claim these are the only solutions. Suppose not; then there are $a, b > 0$ with $f(a) \neq a$ and $f(b) \neq 1/b$. We set $p = a$, $q = b$, and $r = s = \sqrt{ab}$ and obtain $(a^{-2} + b^2)/2f(ab) = (a^2 + b^2)/2ab$, or

$$f(ab) = \frac{ab(a^{-2} + b^2)}{a^2 + b^2}.$$

However, we know that $f(ab) = ab$ or $1/ab$. In the first case, $a^2 + b^2 = a^{-2} + b^2$, so $a = 1$ and $f(1) = 1$ contradicts our assumption on a. Likewise, if $f(ab) = 1/ab$, then $a^2b^2(a^{-2} + b^2) = a^2 + b^2$, so that $b = 1$, again a contradiction. We conclude that $f(x) = x, 1/x$ are the only solutions.

13. Consider those functions $f : \mathbb{N} \to \mathbb{N}$ (here $\mathbb{N}$ denotes the positive integers) which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2009)$.

Solution. First notice that $f(m+n) \geq f(m) + f(f(n)) - 1 \geq f(m)$, so f is nondecreasing.

We claim that $f(n) \leq n+1$. To the contrary, suppose that $f(n) = m+n$ where $m > 1$. We write

$$\begin{aligned} f(2n) &\geq f(n) + f(f(n)) - 1 \geq 2(m+n) - 1 = 2(m+n-1) + 1, \\ f(4n) &\geq f(2n) + f(f(2n)) - 1 \geq 2(m+n-1) + 1 + 2(m+n-1) = 4(m+n-1) + 1, \\ &\vdots \\ f(2^k n) &\geq 2f(2^{k-1}n) - 1 \geq 2^k(m+n-1) + 1. \end{aligned}$$

Notice that $f(k+1) \geq f(1) + f(f(k)) - 1 \geq f(f(k))$, so that

$$f(2^k n + 1) \geq f(f(2^k n)) \geq f(2^k(m+n-1) + 1),$$

and so

$$f(2^k n + 1) = f(2^k n + 1) = f(2^k n + 2) = \cdots = f(2^k(m+n-1) + 1).$$

For some k we have $2^k(m-1) \geq n$. Then

$$\begin{aligned} f(2^k n + 1) &= f(2^k(m+n-1) + 1) \\ &\geq f(2^k(m+n-1) + 1 - n) + f(f(n)) - 1 \\ &= f(2^k n + 1) + f(f(n)) - 1 \\ &\geq f(2^k n + 1) + m + n - 1, \end{aligned}$$

so $m + n \leq 1$, a contradiction. This proves the claim.

We prove that any value from 1 to 2010 may be obtained by $f(2009)$. Indeed, for any value less than or equal to 2009, we may choose a real number $0 < \alpha \leq 1$ and set $f(n) = \lfloor n\alpha \rfloor$, because

$$f(m+n) = \lfloor (m+n)\alpha \rfloor \geq \lfloor m\alpha \rfloor + \lfloor n\alpha \rfloor > \lfloor m\alpha \rfloor + \lfloor \lfloor n\alpha \rfloor \alpha \rfloor - 1 = f(m) + f(f(n)) - 1.$$

To obtain $f(2009) = 2010$, consider

$$f(n) = \begin{cases} n, & 2009 \nmid n \\ n+1, & 2009 | n \end{cases}$$

Then $2009 \nmid f(n)$, so $f(f(n)) = f(n)$. Then $f(m+n) \geq f(m) + f(n) - 1$ because $f(m+n) \geq m+n$ and if $f(m) + f(n) - 1 > m+n$ then 2009 divides both m and n and $f(m+n) = m+n+1 = f(m) + f(n) - 1$. Thus any value from 1 to 2010 may be achieved by such functions.

14. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Then
$$f(x) = \left(x + \frac{a_{n-1}}{n}\right)^n + g(x)$$
for some polynomial $g(x)$ of degree at most $n-2$. For large enough x,
$$0 \le |g(x)| < \left(x + \frac{a_{n-1}-1}{n}\right)^n - \left(x + \frac{a_{n-1}}{n}\right)^n$$
since the RHS has degree $n-1$. Then
$$\left(x + \frac{a_{n-1}-1}{n}\right)^n < f(x) < \left(x + \frac{a_{n-1}+1}{n}\right)^n$$
for large enough x so we must have $f(x) = \left(x + \frac{a_{n-1}}{n}\right)^n$ for large enough x, and this must be true for all x. Hence $f(x) = (x+c)^n$ for some integer c.

15. From (a) it follows that $f(xf(x)) = xf(x)$ for all $x > 0$. By induction on n, we have that if $f(a) = a$ for some $a > 0$, then $f(a^n) = a^n$ for all $n \in \mathbb{N}$. Note also that $a \le 1$, since otherwise
$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} a^n = \infty,$$
in contradiction to (b).

On the other hand, $a = f(1 \cdot a) = f(1 \cdot f(a)) = af(1)$. Hence
$$1 = f(1) = f(a^{-1}a) = f(a^{-1}f(a)) = af(a^{-1}),$$
implying $f(a^{-1}) = a^{-1}$. Thus we have (as above) $f(a^{-n}) = a^{-n}$ for all $n \in \mathbb{N}$ and $a^{-1} \le 1$. In conclusion, the only $a > 0$ such that $f(a) = a$ is $a = 1$. Hence the identity $f(xf(x))$ implies $f(x) = \frac{1}{x}$ for all $x > 0$. It is easy to check that this function satisfies (a) and (b) of the problem.

16. Yes. We verify that $f(n) = \left\lfloor \frac{1+\sqrt{5}}{2}n + \frac{1}{2} \right\rfloor$ is a function with all the required properties. We can compute $f(1) = 2$, and note that $\lfloor x \rfloor < \lfloor x+1 \rfloor$ and $\frac{1+\sqrt{5}}{2} > 1$ imply that $f(n) < f(n+1)$.

Now we verify the second part. Let $c = \frac{1+\sqrt{5}}{2}$. Noting that $c > 1$, we have
$$cn + \frac{c}{2} > cn + \frac{1}{2} \ge \left\lfloor cn + \frac{1}{2} \right\rfloor > cn - \frac{1}{2} > cn - \frac{c}{2}.$$
Multiplying by $\frac{1}{c} = \frac{\sqrt{5}-1}{2}$ we get
$$n + \frac{1}{2} > \frac{\sqrt{5}-1}{2}\left\lfloor cn + \frac{1}{2} \right\rfloor > n - \frac{1}{2}.$$
Adding $\left\lfloor cn + \frac{1}{2} \right\rfloor + \frac{1}{2}$ we get
$$\left\lfloor cn + \frac{1}{2} \right\rfloor + n + 1 > c\left\lfloor cn + \frac{1}{2} \right\rfloor + \frac{1}{2} > \left\lfloor cn + \frac{1}{2} \right\rfloor + n.$$
Thus
$$\left\lfloor cn + \frac{1}{2} \right\rfloor + n + 1 > \left\lfloor c\left\lfloor cn + \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rfloor \ge \left\lfloor cn + \frac{1}{2} \right\rfloor + n.$$
or $f(n) + n + 1 > f(f(n) \ge f(n) + n$, implying $f(f(n)) = f(n) + n$.