Functional Equations Solutions

Solutions to Problems

1. Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Find all functions $\mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x^2 + y) = f(f(x)) + \frac{f(xy)}{f(x)}.$$

Solution. Suppose $f(x) = x^2$ for all $x$. Then

$$(x^2 + y)^2 = x^4 + \frac{x^2 y^2}{x^2},$$

or $2x^2y = 0$, an abject contradiction. We conclude that there is some $x_0$ with $f(x_0) \neq x_0^2$. Letting $y = f(x_0) - x_0^2$, we obtain

$$f(f(x_0)) = f(f(x_0)) + \frac{f(x_0y)}{f(x_0)},$$

so that the fraction would be zero, contradicting the given range.

2. Find all functions $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ such that

$$f(x) + f \left( \frac{1}{1-x} \right) = 1 + \frac{1}{x(1-x)}.$$

Solution. We employ the facts that

$$\frac{1}{1-x} = 1 - \frac{1}{x} \quad \text{and} \quad 1 - \frac{1}{(1-x)} = x$$

to obtain

$$f \left( \frac{1}{1-x} \right) + f \left( 1 - \frac{1}{x} \right) = 3 - x - \frac{1}{x} \quad \text{and} \quad f \left( 1 - \frac{1}{x} \right) + f(x) = 2 + x - \frac{1}{1-x},$$

which gives us three equations in three unknowns. Subtracting the second equation from the third, we obtain

$$f(x) - f \left( \frac{1}{1-x} \right) = 2x - 1 + \frac{1}{x} - \frac{1}{1-x}.$$

Finally, adding this to the first and dividing by 2 we get

$$f(x) = x + \frac{1}{x},$$

the only solution.
3. Find all functions \( f : \mathbb{Z} \to \mathbb{Z} \) such that for all \( x, y \in \mathbb{Z} \),
\[ f(x - y + f(y)) = f(x) + f(y). \]

**Solution.** We will define \( g(x) = f(x) - x \), so that our equation becomes
\[ g(x + g(y)) = g(x) + y. \]

Fixing \( x \) and letting \( y \) run through the integers we see that \( g \) is surjective. Let \( t \) be such that \( g(t) = 0 \); then
\[ g(x) = g(x + g(t)) = g(x) + t, \]
so \( t = 0 \) and \( g(0) = 0 \). We may also find \( k \) with \( g(k) = 1 \). Setting \( y \) to be this \( k \) we obtain
\[ g(x + 1) = g(x) + k \]
for any \( x \), so that \( g(x) = kx \) for all \( x \). Surjectivity forces \( k = \pm 1 \), and we immediately see that \( g(x) = \pm x \) (or \( f(x) = 0 \) or \( 2x \)) are the only solutions.

4. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that
\[ f(f(x) + y) = 2x + f(f(y) - x) \]
for all real \( x \) and \( y \).

**Solution.** Setting \( y = -f(x) \) we obtain
\[ f(0) = 2x + f(f(-f(x)) - x), \]
so that as we run \( x \) through all the reals we obtain surjectivity for \( f(x) \). Let \( r \) be such that \( f(r) = 0 \); then
\[ f(y) = f(f(r) + y) = 2r + f(f(y) - r), \]
or \( f(f(y) - r) = (f(y) - r) - r. \)

However \( f(y) - r \) runs through all the reals, so for any \( x \) we may say \( f(x) = x - r \), and all such solutions work in the original equation.

5. Determine all functions \( f : \mathbb{R} \to \mathbb{R} \) such that
\[ f((x + y)^2) = (x + y)(f(x) + f(y)). \]

**Solution.** Setting \( y = 0 \) we obtain \( f(x^2) = xf(x) \), so \( f(0) = 0 \) and also
\[ (x + y)(f(x) + f(y)) = f((x + y)^2) = (x + y)f(x + y). \]

If \( x + y \neq 0 \) this immediately gives Cauchy’s equation \( f(x) + f(y) = f(x + y) \). If \( x + y = 0 \) with \( x \neq 0 \), then
\[ xf(x) = f(x^2) = (-x)f(-x) \]
so that
\[ f(x) + f(-x) = 0 = f(0); \]
the case \( x = y = 0 \) is clear. We conclude that for any \( x, y \),
\[
(x + y)(f(x) + f(y)) = f((x + y)^2) = f(x^2 + 2xy + y^2) = f(x^2) + f(2xy) + f(y^2) = xf(x) + f(2xy) + yf(y),
\]
so that
\[ f(2xy) = xf(y) + yf(x). \]
Setting \( y = 1 \) we obtain
\[ 2f(x) = f(2x) = xf(1) + f(x), \]
so \( f(x) = kx \) where \( k = f(1) \), and these are all solutions.

6. Let \( \mathbb{N} = \{0, 1, 2, \ldots \} \) be the set of nonnegative integers. Determine whether or not there exists a bijective function \( f : \mathbb{N} \to \mathbb{N} \) such that for each \( m, n \in \mathbb{N} \),
\[ f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n). \]

**Solution.** There exist infinitely (even uncountably) many such.

Let us rewrite the given equation as
\[ f\left(\frac{(3m+1)(3n+1) - 1}{3}\right) = \frac{(4f(m)+1)(4f(n)+1) - 1}{4}. \]

Then what we seek is a bijection \( g : S \to T \) where \( S \) is the set of \( 3k+1 \) integers and \( T \) is the set of \( 4k+1 \) integers, such that
\[ g(xy) = g(x)g(y). \]

Let \( \Pi_{a,m} \) denote the set of primes congruent to \( a \) mod \( m \). It is an elementary fact that \( \Pi_{1,3}, \Pi_{2,3}, \Pi_{1,4}, \) and \( \Pi_{3,4} \) are all infinite, so we may choose bijections \( \Pi_{1,3} \to \Pi_{1,4} \) and \( \Pi_{2,3} \to \Pi_{3,4} \). From these we may define a function \( g : S \to T \) mapping prime factorizations to their image under the two bijections. This works because an integer is of the form \( 3k+1 \) if and only if it is factored into some number of \( 3k+1 \) primes and an even number of \( 3k+2 \) primes, and similarly for an integer of the form \( 4k+1 \). Therefore we have obtained our solution (and the uncountably many bijections between the sets of primes yield the claimed uncountably many solutions).

7. Find all functions \( f : \mathbb{R}^+ \to \mathbb{R} \) satisfying
\[ f(x) + f(y) \leq \frac{f(x+y)}{2} \quad \text{and} \quad \frac{x}{f(x)} + \frac{y}{f(y)} \geq \frac{x+y}{x+y}, \]
for all \( x, y > 0 \).

**Solution.** To avoid confusion we set \( g(x) = -f(x)/x \), and look for solutions to
\[ xg(x) + yg(y) \geq \left(\frac{x+y}{2}\right) g(x+y) \] and \( g(x) + g(y) \leq g(x+y) \).
Substituting \( y = x \) we obtain
\[
2xg(x) \geq xg(2x) \text{ and } 2g(x) \leq g(2x),
\]
so that \( g(2x) = 2g(x) \). From this we obtain \( g(2^n x) = 2^n g(x) \) for any \( x \). What’s more,
\[
g(nx) \geq g(x) + g((n-1)x) \geq 2g(x) + g((n-2)x) \geq \cdots \geq ng(x)
\]
for any positive integer \( n \). Additionally,
\[
5xg(x) = xg(x) + 2xg(2x) \geq \frac{3}{2} xg(3x) \geq \frac{9}{2} xg(x),
\]
so \( 10g(x) \geq 9g(x) \) and \( g(x) \geq 0 \) for all \( x \). Finally, whenever \( y > x \) we have \( g(y - x) \leq g(y) \), and so \( g(x) \) is nondecreasing.

Let \( x \) be a positive real. Suppose we have positive integers \( n, n', m, m' \) with
\[
\frac{2n'}{m'} \geq x \geq \frac{m}{2^n}.
\]
Then
\[
\frac{2n'}{m} g(1) = \frac{1}{m'} g \left( \frac{2n'}{m} \right) \geq g \left( \frac{2n'}{m'} \right) \geq g(x) \geq g \left( \frac{m}{2^n} \right) \geq mg \left( \frac{1}{2^n} \right) = m g(1).
\]
Since these fractions may be chosen arbitrarily close to one another, we conclude that \( g(x) = g(1)x \). The given inequalities work as well; the first is Titu’s Lemma and the second is an equality. Thus \( f(x) = -ax^2 \) for \( a \geq 0 \) are the only solutions.

8. Let \( \mathbb{R} \) denote the set of real numbers. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that
\[
f(x + y) + f(x)f(y) = f(xy) + 2xy + 1.
\]

**Solution.** Setting \( x = y = 0 \), we obtain \( f(0)^2 = 1 \), if \( f(0) = \pm 1 \). However, if \( f(0) = 1 \) then we may take \( y = 0 \) and obtain \( 2f(x) = f(x) + 1 \), or \( f(x) = 1 \) for all \( x \), a contradiction. Therefore we conclude that \( f(0) = -1 \).

Next, we look at \( (x, y) = (-1, 1) \), obtaining
\[
-1 + f(1)f(-1) = f(-1) - 1,
\]
or
\[
f(-1)(f(1) - 1) = 0.
\]

**Case (a).** If \( f(1) = 1 \) then using \( (x, y) = (x - 1, 1) \) we obtain
\[
f(x) + f(x - 1) = f(x - 1) + 2x - 1,
\]
or \( f(x) = 2x - 1 \), our first solution.
Case (b). If $f(-1) = 0$, we may take $(x, y) = (-1, -1)$ and get $f(-2) = f(1) + 3$, and then with $(x, y) = (-2, 1)$ we get

$$f(-2)f(1) = f(-2) - 3.$$ 

Combining these, we obtain

$$f(-2)(f(1) - 1) = 0.$$ 

Case (bi). Suppose $f(1) = 0$. We obtain

$$f(x) = f((x - 1) + 1) = f(x - 1) + 2x - 1 = f(-x) - 2x + 1 + 2x - 1 = f(-x),$$

so that $f(x)$ is an even function. Finally, taking $(x, y) = (x, \pm x)$, we get

$$-1 + f(x)^2 = f(x - x) + f(x)f(-x) = f(-x^2) - 2x^2 + 1$$

and

$$f(2x) + f(x)^2 = f(x^2) + 2x^2 + 1 = f(-x^2) + 2x^2 + 1,$$

so we get $f(2x) = 4x^2 - 1, f(x) = x^2 - 1$, a solution.

Case (bii). Suppose $f(1) = -2$. In this case we get

$$f(x + 1) - 2f(x) = f(x) + 2x + 1, \text{ or } f(x) = 3f(x - 1) + 2x - 1.$$ 

As $f(-1) = 0$, we have

$$f(x) = 3f(-x) - 2x + 1 + 2x - 1 = 3f(-x) - 4x + 2.$$ 

Swapping the roles of $x$ and $-x$ here, we get $f(-x) = 3f(x) + 4x + 2$, a system of two equations in two unknowns that we may use to solve for $f(x)$, getting $f(x) = -x - 1$, the third solution.

9. Let $\mathbb{R}^+$ denote the set of positive real numbers and let $k \in \mathbb{R}^+$ be a constant. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)f(y) = kf(x + yf(x))$$

for all positive real numbers $x$ and $y$.

Solution. Suppose that $f(x)$ is injective. Setting $x = a$ and $y = 1$, we obtain

$$f(a)f(1) = kf(a + f(a)).$$

Setting $x = 1$ and $y = a$, we get

$$f(1)f(a) = kf(1 + af(1)).$$

We conclude by injectivity that $a + f(a) = 1 + af(1)$, or

$$f(a) = 1 - a(f(1) - a),$$

where $f(x)$ is an odd function and $f(1) = 1$. Thus $f(x)$ is of the form $f(x) = 1 - x$. Since $f(x)$ is injective, we must have $f(1) = 1$. Therefore, $f(x) = 1 - x$ is a solution.
or in other words \( f(x) = 1 + cx \) for some constant \( c \). In order for this to be a solution, we need

\[
(1 + cx)(1 + cy) = k(1 + c(x + y(1 + cx))) = k(1 + cx)(1 + cy).
\]

Therefore this is a solution if and only if \( k = 1 \).

Now suppose that \( f(x) \) is not injective, so \( f(a) = f(b) = c \) for some \( a < b \in \mathbb{R}^+ \). We claim that \( f(x) = c \) for all positive \( x \).

First, for all \( y \) and \( f \) we have

\[
f(a)f(y) = kf(a + yf(a)) = kf(a + cy)
\]

and

\[
f(b)f(y) = kf(b + yf(b)) = kf(b + cy),
\]

so

\[
f((a - b) + b + cy) = f(a + cy) = f(b + cy).
\]

We conclude that \( f(x) \) is periodic of period \( a - b \) for all \( y \geq b \).

Now suppose we have \( x_1 \) and \( x_2 \) with \( f(x_1) > f(x_2) \). We conclude that for any \( y \), \( f(x_1)f(y) \neq f(x_2)f(y) \). However we may choose \( y \) so large that both \( x_1 + yf(x_1) \) and \( x_2 + yf(x_2) \) are greater than or equal to \( b \), and also that

\[
[x_1 + yf(x_1)] - [x_2 + yf(x_2)] = (x_1 - x_2) + y(f(x_1) - f(x_2)) = n(a - b)
\]

for some positive integer \( n \), so that in fact

\[
f(x_1)f(y) = f(x_1 + yf(x_1)) = f(x_2 + yf(x_2)) = f(x_2)f(y),
\]

a contradiction so that in fact \( f(x) \) is a constant. Finally, if \( f(x) = c \) then \( c^2 = kc \) shows that \( f(x) = k \). These families comprise all solutions.

10. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
f(f(x + f(y)) - 1) = f(x) + f(x + y) - x.
\]

**Solution.** Let us first prove that \( f(x) \) is injective. If \( f(y) = f(y') \) for some \( y \neq y' \), we have

\[
f(x) + f(x + y) - x = f(f(x + f(y)) - 1) = f(f(x + f(y')) - 1) = f(x) + f(x + y') - x,
\]

so that \( f(x + y) = f(x + y') \) and \( f \) is periodic with period \( y - y' = p \). However,

\[
f(x) + f(x + y) - x = f(f(x + f(y)) - 1) = f(f(x + p + f(y)) - 1) = f(x + p) + f(x + p + y) - x - p = f(x) + f(x + y) - x - p,
\]

which is a contradiction.
so \( p = 0 \), a contradiction so that \( f \) is injective.

Now let \( y \) be arbitrary and set \( x = f(y) - f(0) \). We obtain

\[
\begin{align*}
f(x) + f(x + 0) - x &= f(f(x + f(0)) - 1) \\
&= f(f(0 + f(y)) - 1) \\
&= f(0) + f(0 + y) - 0 \\
&= f(0) + f(0) + x,
\end{align*}
\]

so that

\[
f(y) = f(0) + x = f(x).
\]

By injectivity, \( x = y \), and so \( f(y) = y + f(0) = y + c \). Now we plug back into our original equation with \( x = y = 0 \):

\[
2c = f(f(c) - 1) = f(2c - 1) = 3c - 1,
\]

so \( c = 1 \) and \( f(x) = x + 1 \) is the only solution.

11. Determine all functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
f(xf(y)) = (1 - y)f(xy) + x^2y^2f(y)
\]

for all real numbers \( x \) and \( y \).

**Solution.** We observe that \( f(x) = 0 \) is a solution, and assume that \( f(x) \) is not identically 0 from now on. Setting \( x = 1 \) we get

\[
f(f(y)) = (1 - y)f(y) + y^2f(y) = (1 - y + y^2)f(y),
\]

and setting \( y = 1 \), we obtain \( f(xf(1)) = x^2f(1) \). If \( f(1) \neq 0 \), we are forced to have \( f(x) = x^2/f(1) \) for all \( x \), so \( f(1) = 1/f(1) \) and \( f(x) = \pm x^2 \). This directly contradicts our first equation:

\[
\pm y^4 = (1 - y + y^2)(\pm y^2).
\]

We conclude that \( f(1) = 0 \). Setting \( y = 1 \) in the first equation, we get

\[
f(0) = f(1) = 0,
\]

so \( f(0) = 0 \).

We claim these are the only two values of 0. Indeed, if \( f(y) = 0 \), then we have

\[
0 = f(xf(y)) = (1 - y)f(xy) + x^2y^2f(y) = (1 - y)f(xy),
\]

so that either \( y = 0, 1 \) or \( f(x) \) is identically 0, contradicting our assumptions.

Next we use \( y = 1/x \).

\[
f(xf(1/x)) = (1 - 1/x)f(1) + f(1/x) = f(1/x),
\]

so for any \( x \neq 0 \) we obtain \( f(f(x)/x) = f(x) \).
Suppose now we have two values \( f(x) = f(y) \neq 0 \). We argue

\[
(1 - x + x^2)f(x) = f(f(x)) = f(f(y)) = (1 - y + y^2)f(y) = (1 - y + y^2)f(x).
\]

So \( y = x, 1 - x \). However then we have

\[
\frac{f(x)}{x} = x, x - 1,
\]

so for each \( x \) either \( f(x) = x^2 \) or \( f(x) = x - x^2 \). The latter case is a solution if it holds for all \( x \).

Suppose now that some \( f(x) = x^2 \). We know

\[
f(x^2) = f(f(x)) = (1 - x + x^2)f(x) = x^2 - x^3 + x^4.
\]

If \( f(x^2) = x^4 \), then \( x^2 - x^3 = 0 \) in which case \( x = 0, 1 \). If \( f(x^2) = x^2 - x^4 \), then \( 2x^4 - x^3 = 0 \), so \( x = 0, \frac{1}{2} \). We already know that \( f(1) = 0 \), and moreover \( 0^2 = 0 - 0^2 \) and \( (\frac{1}{2})^2 = \frac{1}{2} - (\frac{1}{2})^2 \), so actually we may conclude that \( f(x) = x - x^2 \) in any case. Thus (in the nonzero case) this is the only solution.

12. Find all functions \( f : (0, \infty) \to (0, \infty) \) such that

\[
\frac{f(p)^2 + f(q)^2}{f(r)^2 + f(s)^2} = \frac{p^2 + q^2}{r^2 + s^2}
\]

for all \( p, q, r, s > 0 \) with \( pq = rs \).

**Solution.** Setting \( p = q = r = s = 1 \) we obtain \( f(1)^2 = f(1) \) and so \( f(1) = 1 \). Now let \( x > 0 \) and \( p = x, q = 1, r = s = \sqrt{x} \) to obtain

\[
\frac{f(x^2) + 1}{2f(x)} = \frac{x^2 + 1}{2x}.
\]

This rearranges into

\[
xf(x)^2 + x = x^2 f(x) + f(x),
\]

or

\[
(xf(x) - 1)(f(x) - x) = 0.
\]

Therefore either \( f(x) = x \) or \( f(x) = 1/x \) for every \( x > 0 \).

The functions \( f(x) = x \) and \( f(x) = 1/x \) both satisfy the conditions of the problem; we claim these are the only solutions. Suppose not; then there are \( a, b > 0 \) with \( f(a) \neq a \) and \( f(b) \neq 1/b \). We set \( p = a, q = b, \) and \( r = s = \sqrt{ab} \) and obtain \( (a^2 - b^2)/2f(ab) = (a^2 + b^2)/2ab \), or

\[
f(ab) = \frac{ab(a^2 + b^2)}{a^2 + b^2}.
\]

However, we know that \( f(ab) = ab \) or \( 1/ab \). In the first case, \( a^2 + b^2 = a^{-2} + b^2 \), so \( a = 1 \) and \( f(1) = 1 \) contradicts our assumption on \( a \). Likewise, if \( f(ab) = 1/ab \), then \( a^2b^2(a^{-2} + b^2) = a^2 + b^2 \), so that \( b = 1 \), again a contradiction. We conclude that \( f(x) = x, 1/x \) are the only solutions.
13. Consider those functions \( f : \mathbb{N} \to \mathbb{N} \) (here \( \mathbb{N} \) denotes the positive integers) which satisfy the condition
\[
f(m + n) \geq f(m) + f(f(n)) - 1
\]
for all \( m, n \in \mathbb{N} \). Find all possible values of \( f(2009) \).

**Solution.** First notice that \( f(m + n) \geq f(m) + f(f(n)) - 1 \geq f(m) \), so \( f \) is nondecreasing.

We claim that \( f(n) \leq n + 1 \). To the contrary, suppose that \( f(n) = m + n \) where \( m > 1 \). We write
\[
\begin{align*}
f(2n) & \geq f(n) + f(f(n)) - 1 \geq 2(m + n) - 1 = 2(m + n - 1) + 1, \\
f(4n) & \geq f(2n) + f(f(2n)) - 1 \geq 2(m + n - 1) + 1 + 2(m + n - 1) = 4(m + n - 1) + 1, \\
& \vdots \\
f(2^k n) & \geq 2f(2^{k-1} n) - 1 \geq 2^k (m + n - 1) + 1.
\end{align*}
\]
Notice that \( f(k + 1) \geq f(1) + f(f(k)) - 1 \geq f(f(k)) \), so that
\[
f(2^k n + 1) \geq f(f(2^k n)) \geq f(2^k (m + n - 1) + 1),
\]
and so
\[
f(2^k n + 1) = f(2^k n + 1) = f(2^k n + 2) = \cdots = f(2^k (m + n - 1) + 1).
\]
For some \( k \) we have \( 2^k (m - 1) \geq n \). Then
\[
\begin{align*}
f(2^k n + 1) & = f(2^k (m + n - 1) + 1) \\
& \geq f(2^k (m + n - 1) + 1 - n) + f(f(n)) - 1 \\
& = f(2^k n + 1) + f(f(n)) - 1 \\
& \geq f(2^k n + 1) + m + n - 1,
\end{align*}
\]
so \( m + n \leq 1 \), a contradiction. This proves the claim.

We prove that any value from 1 to 2010 may be obtained by \( f(2009) \). Indeed, for any value less than or equal to 2009, we may choose a real number \( 0 < \alpha \leq 1 \) and set \( f(n) = \lfloor n \alpha \rfloor \), because
\[
f(m + n) = \lfloor (m + n) \alpha \rfloor \geq \lfloor m \alpha \rfloor + \lfloor n \alpha \rfloor > \lfloor m \alpha \rfloor + \lfloor \lfloor n \alpha \rfloor \alpha \rfloor - 1 = f(m) + f(f(n)) - 1.
\]

To obtain \( f(2009) = 2010 \), consider
\[
f(n) = \begin{cases} 
n, & 2009 \mid n \\
n + 1, & 2009 \nmid n 
\end{cases}
\]
Then \( 2009 \mid f(n) \), so \( f(f(n)) = f(n) \). Then \( f(m + n) \geq f(m) + f(n) - 1 \) because \( f(m + n) \geq m + n \) and if \( f(m) + f(n) - 1 > m + n \) then \( 2009 \) divides both \( m \) and \( n \) and \( f(m + n) = m + n + 1 = f(m) + f(n) - 1 \). Thus any value from 1 to 2010 may be achieved by such functions.
14. Suppose \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0. \) Then
\[
f(x) = \left(x + \frac{a_{n-1}}{n}\right)^n + g(x)
\]
for some polynomial \( g(x) \) of degree at most \( n - 2 \). For large enough \( x \),
\[
0 \leq |g(x)| < \left(x + \frac{a_{n-1} - 1}{n}\right)^n - \left(x + \frac{a_{n-1}}{n}\right)^n
\]
since the RHS has degree \( n - 1 \). Then
\[
\left(x + \frac{a_{n-1} - 1}{n}\right)^n < f(x) < \left(x + \frac{a_{n-1} + 1}{n}\right)^n
\]
for large enough \( x \) so we must have \( f(x) = \left(x + \frac{a_{n-1}}{n}\right)^n \) for large enough \( x \), and this must be true for all \( x \). Hence \( f(x) = (x+c)^n \) for some integer \( c \).

15. From (a) it follows that \( f(xf(x)) = xf(x) \) for all \( x > 0 \). By induction on \( n \), we have that if \( f(a) = a \) for some \( a > 0 \), then \( f(a^n) = a^n \) for all \( n \in \mathbb{N} \). Note also that \( a \leq 1 \), since otherwise
\[
\lim_{n \to \infty} f(a^n) = \lim_{n \to \infty} a^n = \infty,
\]
in contradiction to (b).

On the other hand, \( a = f(1 \cdot a) = f(1 \cdot f(a)) = af(1) \). Hence
\[
1 = f(1) = f(a^{-1}a) = f(a^{-1}f(a)) = af(a^{-1}),
\]
implicating \( f(a^{-1}) = a^{-1} \). Thus we have (as above) \( f(a^{-n}) = a^{-n} \) for all \( n \in \mathbb{N} \) and \( a^{-1} \leq 1 \). In conclusion, the only \( a > 0 \) such that \( f(a) = a \) is \( a = 1 \). Hence the identity \( f(xf(x)) \) implies \( f(x) = \frac{1}{x} \) for all \( x > 0 \). It is easy to check that this function satisfies (a) and (b) of the problem.

16. Yes. We verify that \( f(n) = \left[\frac{1 + \sqrt{5}}{2}n + \frac{1}{2}\right] \) is a function with all the required properties. We can compute \( f(1) = 2 \), and note that \( |x| < |x+1| \) and \( \frac{1 + \sqrt{5}}{2} > 1 \) imply that \( f(n) < f(n+1) \).

Now we verify the second part. Let \( c = \frac{1 + \sqrt{5}}{2} \). Noting that \( c > 1 \), we have
\[
\frac{cn}{2} + \frac{c}{2} > cn + \frac{1}{2} \geq \left[\frac{cn + 1}{2}\right] > cn - \frac{1}{2} > cn - \frac{c}{2}.
\]
Multiplying by \( \frac{1}{c} = \frac{\sqrt{5} - 1}{2} \) we get
\[
n + \frac{1}{2} > \frac{\sqrt{5} - 1}{2} \left[\frac{cn + \frac{1}{2}}{2}\right] > n - \frac{1}{2}.
\]
Adding \( \left[\frac{cn + \frac{1}{2}}{2}\right] + \frac{1}{2} \) we get
\[
\left[\frac{cn + \frac{1}{2}}{2}\right] + n + 1 > c \left[\frac{cn + \frac{1}{2}}{2}\right] + \frac{1}{2} > \left[\frac{cn + \frac{1}{2}}{2}\right] + n.
\]
Thus
\[
\left[\frac{cn + \frac{1}{2}}{2}\right] + n + 1 > \left[\frac{cn + \frac{1}{2}}{2}\right] + \frac{1}{2} > \left[\frac{cn + \frac{1}{2}}{2}\right] + n.
\]
or \( f(n) + n + 1 > f(f(n)) \geq f(n) + n \), implying \( f(f(n)) = f(n) + n \).