The Foundations of Geometry

We'll begin our explorations of geometry with the simplest thing there is, a point.

1 What is a point?

We use the word "point" a lot. We talk about going from point A to point B. We talk about a point in time.

What does a point mean?

Is a point a dot?

No: if so, then a gnat could fit in a point. What if we cut this point in half, then in half again?

But a point should be smaller than a bug. A point should be smaller than the smallest thing that you can think of. A dust speck, an atom. It's so small you can't see it. It's so small that it doesn't make sense to talk about its size.

If you ask mathematicians what a point is, they'll say that a point is an *undefined term*. Because a point seems like the simplest thing that there is; how can we talk about it without talking about more complicated things? How do we even start with geometry?

1.1 What is a Point?

Philosophers, mathematicians, and the common folk have debated the question of "What is a Point?" for centuries. But while Points were easy to talk about, no one could easily get their hands on one. Sages said that they were little black circular creatures, and they also said that if you tried to grab a Point it would slip between you fingers and then pop out of sight.

The Sages claimed to have lots of isolated Points stored in jars in their temples, but they never showed them to normal people. "Points are sacred objects," they said, "they are the essence of purity, and must be kept away from foul air."

Sometimes children would find little black specks that looked like Points when playing in the fields. Typically they were fuzzy and smelled like mothballs. "To tell whether something is a point, see if you can divide it into two," they had learned in class. So one of the boys would run to get a pocket knife while the other kids stood guard over the suspected Point, making sure it didn't blow away.

When the boy jabbed at the Point, though, it split into two parts, each one half of the original point, to the groan of the crowd.

"Well, *that* wasn't a Point," says one of the other boys.

"Try cutting it in half again," says a girl.

So the boy with the knife cut the black speck again, and again it split into two halves.

"Don't anyone move," he said, "I am getting to the bottom of this. If we keep cutting this thing in half, eventually we'll get down to something that can't *be* cut." So he cut again, and again, and finally the specks were so small that he couldn't see them anymore, and the kids were just staring at the dirt.

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This happened again and again, with thousands of different children.

The Sages made up "Axioms" and "Theorems" about Points and the shapes they made up, and called it Geometry. They sent beautifully bound textbooks to the schools. People lived their lives by this book, memorizing Axioms and Theorems about things that the Sages kept locked in their temples, that had never set eyes on.

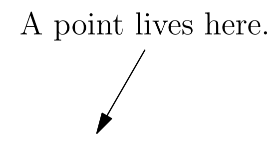
Mathematicians (close friends of the sages) seemed to know everything about Geometry, even though Geometry was built from Points and they said, "Points are undefined objects." If even mathematicians couldn't define Points, what could normal people hope for?

And then you come around. Can you answer the age-old question, what is a point?

Problem 1: What is a point?

1.2 Imagine the Space Around You

Maybe we can't see what a point is, because it's just a single location, with no width, no length and no height.



But you've heard the term "point" and used the term "point". You *know* what a point is, because you can *imagine* it. Geometry starts with your imagination. You start with points, but soon you'll be imagining more than just points.

"The Sages don't *have* lots of Points stocked up in their temples," you say to your friends clustered around a little black speck on the soccer field, "They *imagine* Points. That's why the mathematicians talk about all this Geometry that we normal people don't get. They *imagine* it. And if we *imagine* Points like they do, we can be mathematicians too."

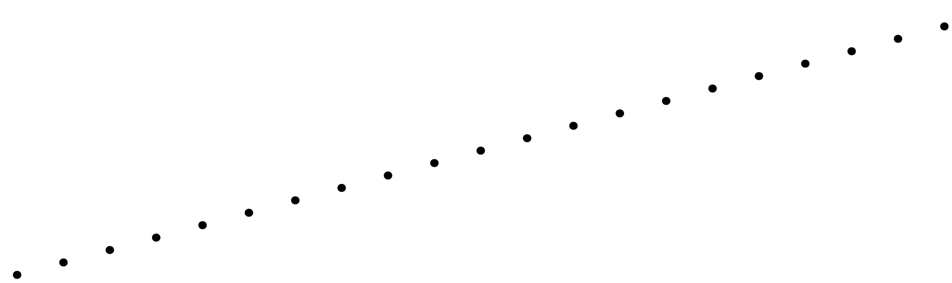
> Geometry starts with your imagination.

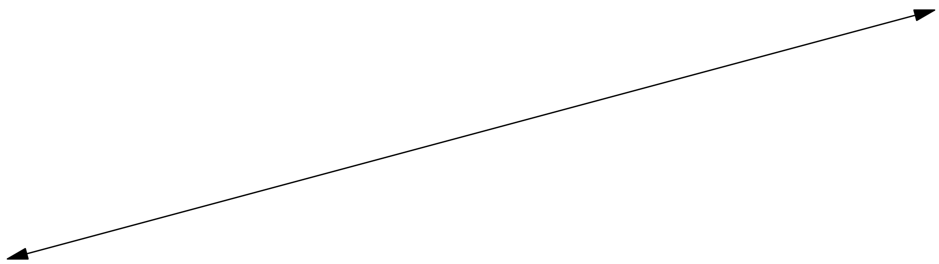
2 More and More Dimensions

A **point** doesn't stretch in *any* direction, so we say it has 0 **dimensions**. We do however, need some way to mark points, so we draw a dot to represent a point, keeping in mind that unlike the dot, the point actually has no length or width.

2.1 Lines

Imagine that we had a whole series of points going in just *one* direction and its opposite, as far as you can see and even more. What would they make? That's right—a **line**.



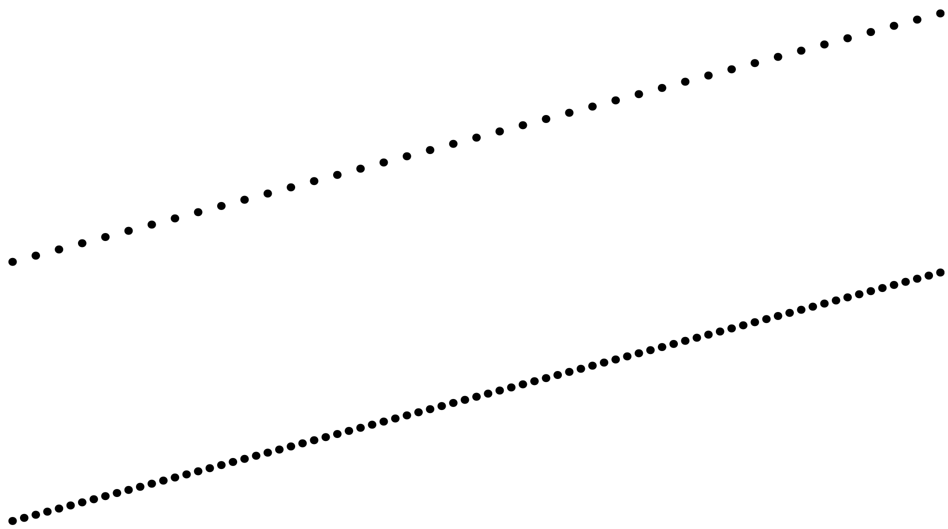


We draw two arrows at the ends to tell you that the line goes on forever. Again, use your imagination, because we can't draw something that goes on forever.

A line has just one direction to it—if you were a point on the line all you can do is move forwards and backwards. So we say a line has one **dimension**. Again, our drawing is an approximation because lines have no width—they are thinner than the lines on notebook paper or the edge of a razor blade.

Problem 2: How many points does a line have?

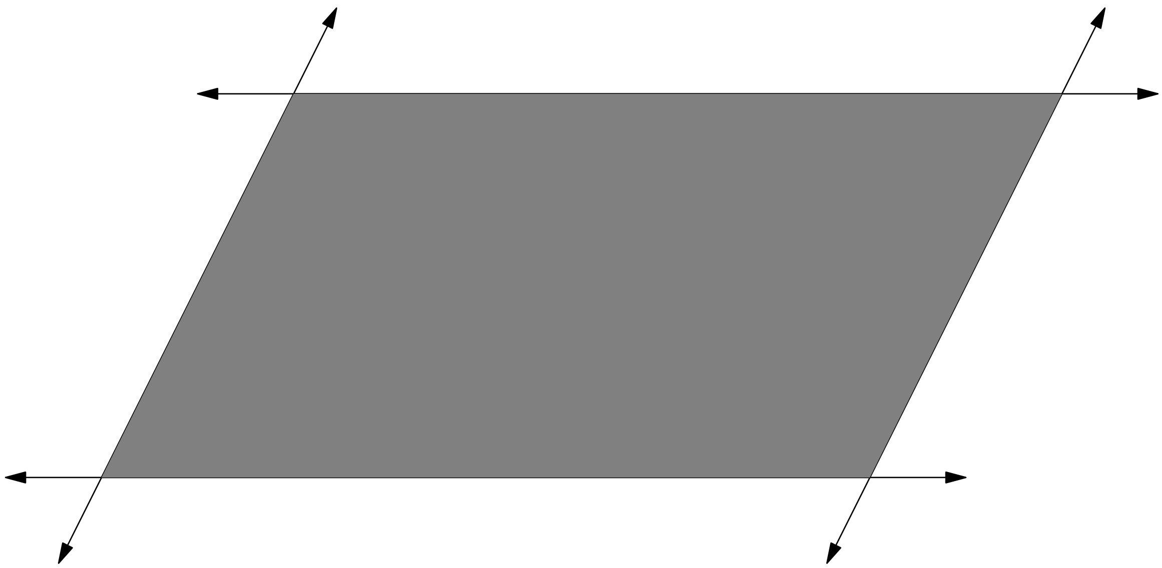
A line has infinitely many points, because for one thing, the line goes on forever. *And*, because points are infinitely small, between every two points you can find another point.



2.2 Planes

Now imagine a restless line moves in another direction:

These lines sweep out a **plane**.



On a plane we have another direction we could move: we can move not just forwards and backwards, but also to the left and right. We say a plane has 2 **dimensions**. Imagine a tabletop or piece of paper that keeps going in every direction, with the same caveat that this piece of paper is infinitely thin.

2.3 Space

Finally, if we add another dimension we are now in the world that you are familiar with: 3-dimensional **space**. You can move not just forwards and backwards, left and right, but also up and down (let's ignore the restrictions of gravity). We won't draw a picture here, because assuming that the universe goes on forever, 3-dimensional space is just the space around you.

We can talk about higher dimensions, but then you *really* have to imagine things. We'll be content with our 3-dimensional world for now, but if you aren't then go ask your neighborhood physicist!

3 How Many Points Do You Need?

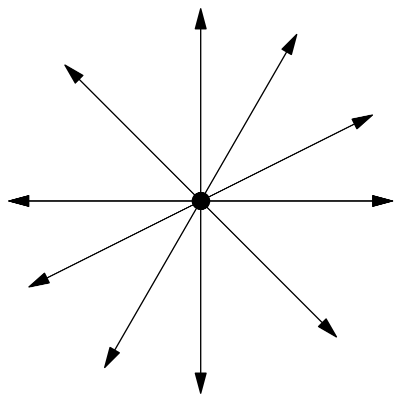
To talk about a line or plane do we need to draw out the entire line or plane? Or is there something easier we could do?

Problem 3: We've seen that a line has infinitely many points. But do you need all of them to know what this line is? How many points of the line do you need to mark out, so that your friend can finish drawing the line for you?

Hint:

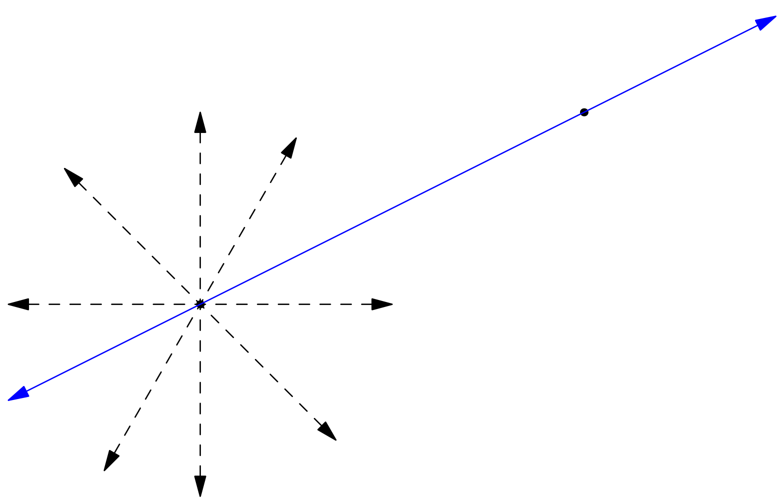
1. Draw a point in the plane. How many lines can you draw through this one point?
2. Draw another point in the plane. How many lines can you draw through these two points?
3. Draw a third point in the plane. How many lines can you draw through all three points?

1. If we just have one point, we can imagine infinitely many lines going through it.

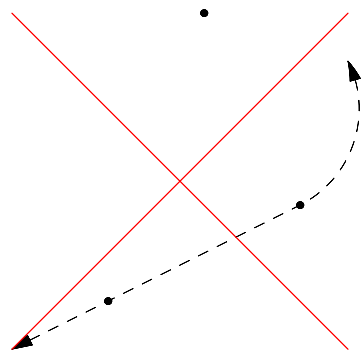


2. Now what if we had two points?

Of all the lines we drew, exactly one goes through both points:



3. If we had three points, then we might be in trouble: we can't find a line through the following three points.



(They could also by chance be on the same line... but this won't be true of *any* three points.)

The fact that two points determine a line is a fundamental fact, called an **axiom**.

Axiom 1: Through any two points, there is exactly one line.

Now that you know two points determine a line, a natural question to ask is what happens if we step up by one dimension?

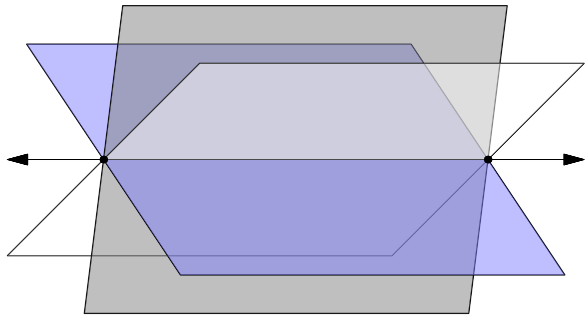
Problem 4: How many points do you need to determine a single plane? Is there any restriction on these points?

Hint: When thinking about the question, remember that a plane is any flat surface going on forever. It could be slanted, like the edge of a roof.

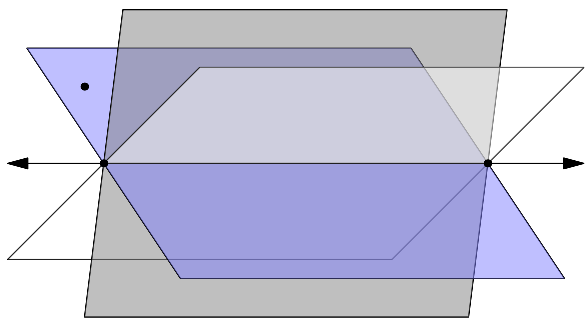
You're going to have to use your imagination a bit more, because you can't draw this problem out on paper. However, feel free to use any props: fingertips, pencil tips, pieces of paper.

- We know we need two points to determine a line, so start with 2 points. Are 2 points enough to determine a plane?
- Now add a 3rd point. How many plane pass through these 3 points?
- What if you added a 4th point?

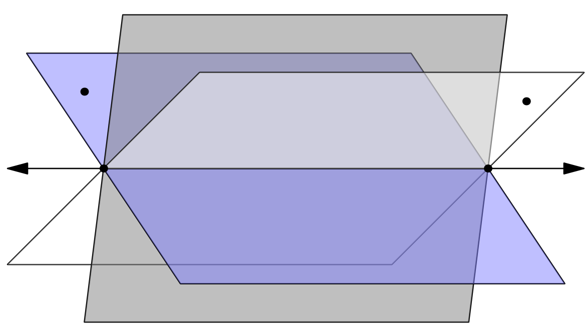
How many points do we need to determine a plane? We know that if we draw two points, we have a line. But there are infinitely many planes going through that line, so infinitely many planes going through the two points.



Let's see what happens when we add a point. If it's on the same line, then any of the planes still works. So let's add a third point *not* on the line. Then there is exactly one plane going through the three points.



If we had four points then we might be in trouble again: we may not be able to find a single plane going through all the points.



Axiom 2: Through any three points not on the same line, there is exactly one plane.

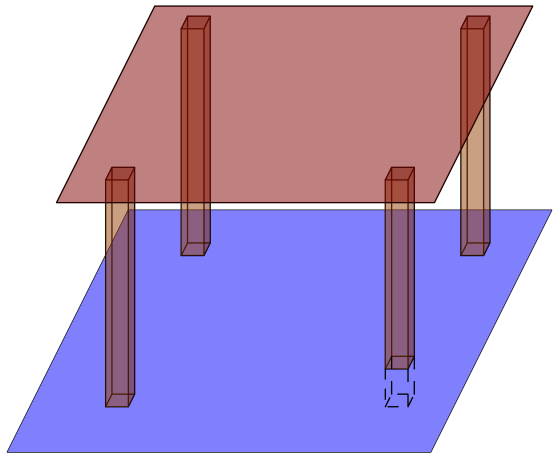
Now let's put our knowledge of points, lines, and planes to use.

Problem 5: A table with four legs sometimes wobbles. Can you explain why? How about a stool with three legs? Is a stool with two legs a good idea?

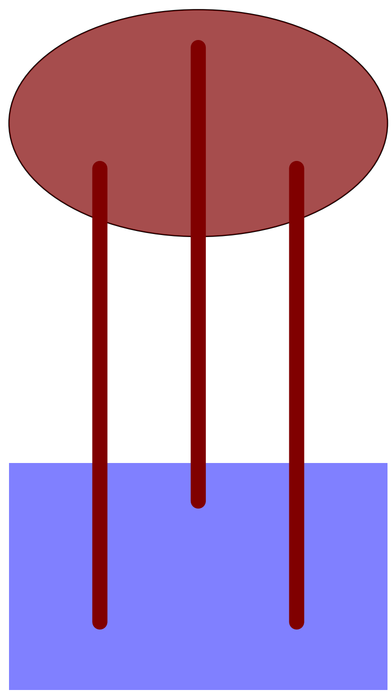
Can you explain why tripods have 3 legs, rather than 4? (Tripods are the stands that you put cameras on.)

What does a table with four legs have to do with anything? Well, we could approximate the tips of the legs as points. The ground is a flat surface—a plane.

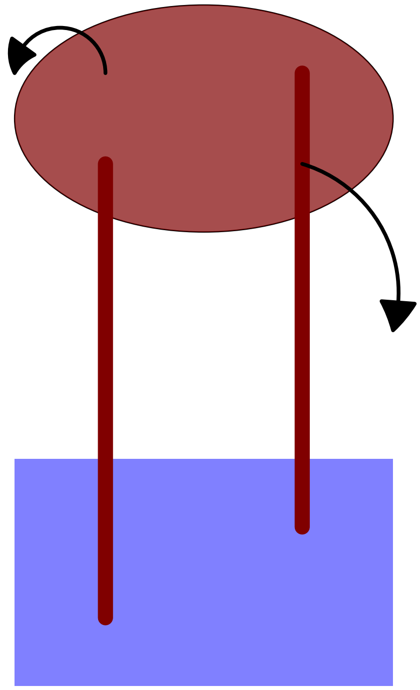
If we have four points, we may not be able to fit them all on the same plane. In this case, we won't be able to get all the table legs to touch the floor at the same time. This happens sometimes for old tables, when one or more of the legs has been worn shorter.



We do know, however, that three points not on a line determine a plane. So for a three-legged stool, there's a unique position for which the ground (plane) goes through the bottoms of all three chair legs (the points).



Finally a stool with two legs is a bad idea, because as we saw there are infinitely many planes going through two points.



A tripod has 3 legs because 3 points uniquely determine a plane. There is exactly one position where all three legs are on the ground, and it will not wobble.

> Although many things in geometry are approximations, we use geometry to model the real world because it often gives accurate answers.

For instance, tips of table legs aren't really points, but the conclusion we drew—4 legs wobbles, 3 legs doesn't—is surprisingly accurate! Geometry is about the real world as much as it is about your imagination.