Parallel Lines Cut Similar Triangles

Thales of Miletos was a Greek philosopher and mathematician who lived in the 6th century BC. Thales was famous for using geometry – especially the geometry of similar triangles that he developed – to solve real-world problems. He was the first person who realized that you could systematically find the height of tall objects using their shadows. That's exactly what Connie and Eric did. Thales used the ideas of similar triangles to find the height of the Great Pyramid and to calculate the distance from ships from shore.

(*If only I were born 2634 years ago, I could have been famous*, Connie thinks, *coming up with similar triangles.*)

Thales's theorems remain the earliest attributed theorems in mathematics. In this section you'll follow in Thales's footsteps yet again as you rediscover an important theorem on similar triangles for yourself.

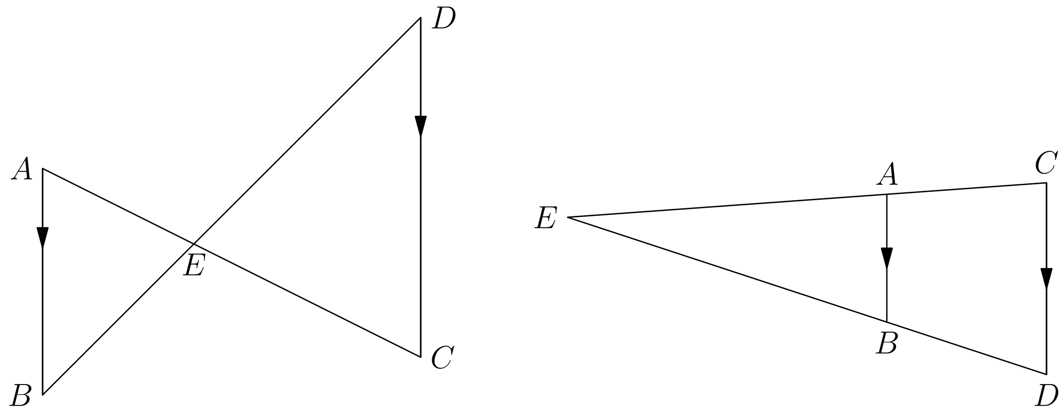
1 Thales's Theorem

One common situation where we can apply similar triangles is when we have parallel lines.

Problem 1: Suppose $\overline{AB} \parallel \overline{CD}$ and $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E. Which triangles are similar and why?

Write all the proportions that follow from similarity.

Note that their are two diagrams that fit this description. The intersection point could be on the line segments themselves, or on the extensions of those line segments.



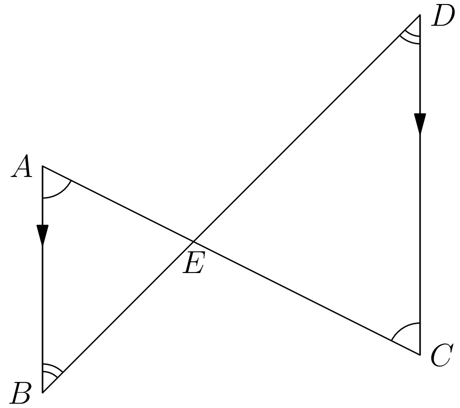
E might be also be on the rays $\overrightarrow{AC}$ and $\overrightarrow{BD}$ instead, but this is essentially the same as the case on the right.

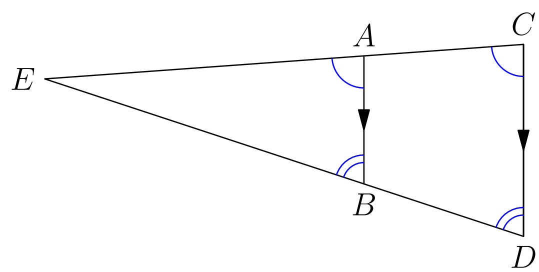
The main idea is that parallel lines give equal angles, and equal angles give similar triangles by AA similarity.

By parallel lines,

$$\angle EAB = \angle ECD, \qquad \angle EBA = \angle EDC.$$

Note that this is true in both cases; the only difference is that when E is on the segments the angles are alternate interior and when E is on the extensions the angles are corresponding.





Then by AA similarity,

$$\triangle EAB \sim \triangle ECD.$$

Parallel lines create equal angles, which give similar triangles.

We get the ratios

$$\frac{EA}{EC} = \frac{EB}{ED} = \frac{AB}{CD}.$$

which we can also express as

$$EA : EB : AB = EC : ED : DC.$$

Theorem 1 (Thales's Theorem)**:** Suppose $\overline{AB} \parallel \overline{CD}$ and $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E. Then

$$\triangle EAB \sim \triangle ECD$$

and we have the proportion

$$EA : EB : AB = EC : ED : CD.$$

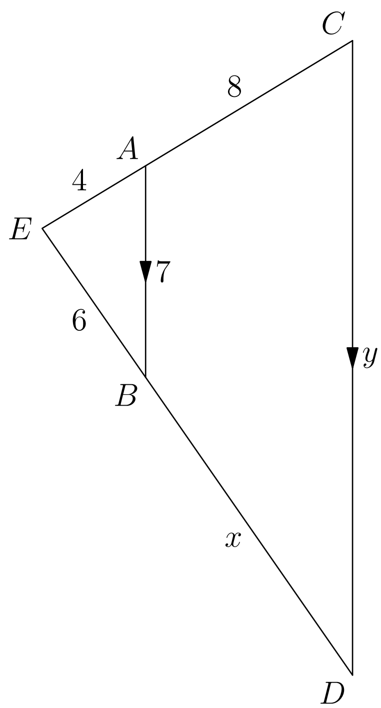
Because the idea behind Thales's Theorem is so important, rather than cite the name, we will often say, "By parallel lines, triangle $\triangle EAB \sim \triangle ACD$." By this we really mean that parallel lines create equal angles, so by AA similarity the triangles are similar.

We have a lot of proportions from parallel lines. But these are not all the proportions we get, as you'll find out if you keep reading.

2 Parallel Lines Cut Proportions I: The Part-to-Whole Principle

Problem 2: Given that $EA = 4$, $AC = 8$ and $EB = 6$, what is BD?

Furthermore, given that $AB = 7$, what is CD?



Let $BD = x$. Unfortunately, AC and BD are not sides in the similar triangles $\triangle EAB$ and $\triangle ECD$! However, $EC = EA + AC$ and $ED = EB + BD$ *are* sides in $\triangle ECD$, and we know everything but BD.

Because $\triangle EAB \sim \triangle ECD$, we get $\frac{EC}{EA} = \frac{ED}{EB}$, or

$$\frac{EA + AC}{EA} = \frac{EB + BD}{EB}.$$

Subtract 1 from both sides:

$$\frac{\cancel{EA} + AC}{EA} = \frac{\cancel{EB} + BD}{EB}.$$

We get that

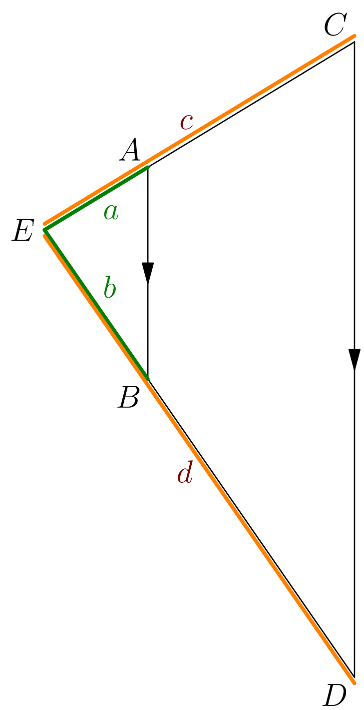
$$\frac{AC}{EA} = \frac{BD}{EB}.$$

Note that we get a ratio involving AC and BD, even though they aren't *side lengths of triangles, by subtracting out $\frac{EA}{EA}$ and $\frac{EB}{EB}$.*

Substituting in our values we get

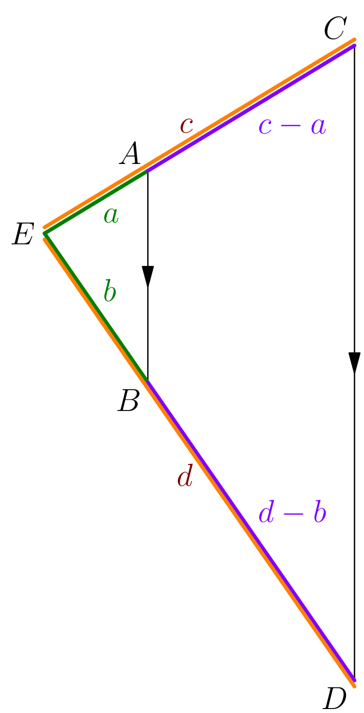
$$\frac{8}{4} = \frac{x}{6} \implies x = \boxed{12}.$$

What we basically did was the following. We know from similar triangles that the lengths of the green and orange segments below are in proportion.



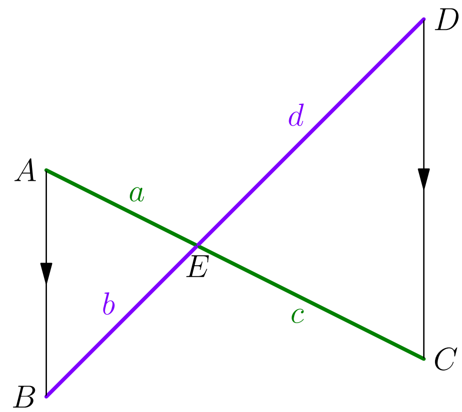
$$a : c \;=\; b : d$$

This means that the lengths of the green, orange, *and purple* segments are all in proportion.



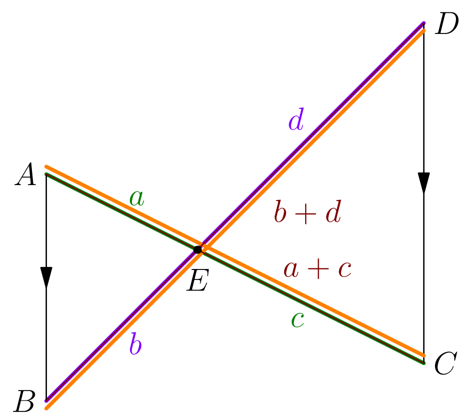
$$a : b : c - a \;=\; b : d : d - b$$

In the same way, if $\overline{AC}$ and $\overline{BD}$ intersected on the inside, we know from similar triangles that the ratios of the green and purple lengths below are in proportion.



$$a : c \; = \; b : d$$

This means that the lengths of the green, purple, *and orange* segments are all in proportion.



$$a : c : a + c \; = \; b : d : b + d$$

Part-to-Whole principle: If $a : c = b : d$ then

$$a : c : (c - a) : (c + a) = b : d : (d - b) : (d + b).$$

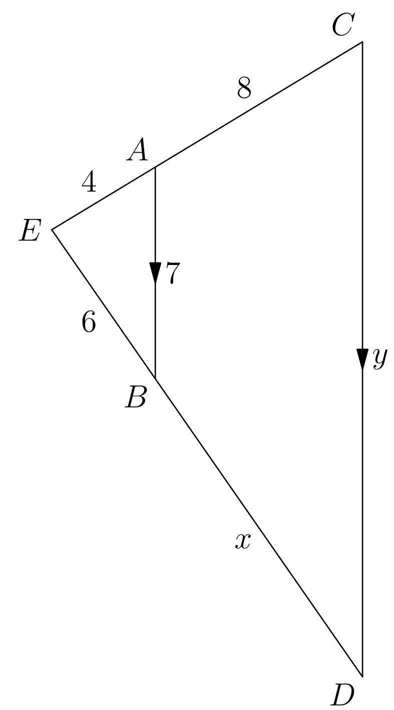
(For the difference we are assuming $b \neq d$.) In other words,if $\frac{a}{b} = \frac{c}{d}$, then

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}.$$

If two pairs of lengths are proportional, then the *difference* and *sum* of those lengths are also proportional.

1. Consider the case where the segments intersect outside. To go from the length of *one part* (the green segments) and length of the whole (the orange segments) to the length of the *other part* (the purple segments), we took the difference.

2. Consider the case where the segments intersect inside. To go from the length of *one part* (the green segments) and the other part (the purple segments) to the length of the *whole* (the orange segments), we took the sum.



Now we find y. Note that

$$\frac{y}{AB} \neq \frac{AC}{AE} = \frac{8}{4} = 2$$

because AC and BD are not sides of our similar triangles! Instead we have by $\triangle EAB \sim \triangle ECB$ that

$$\frac{y}{AB} = \frac{EC}{EA} = \frac{4+8}{4} = 3 \implies y = 3AB = \boxed{21}.$$

Before you move on, make sure you understand why

$$\frac{AC}{EA} = \frac{BD}{EB} \text{ but } \frac{AC}{EA} \neq \frac{CD}{AB}.$$

Call $\triangle EAB$ the "small" triangle and $\triangle ECD$ the "big" triangle. Then AC is a length in the big triangle minus a (corresponding) length in the small triangle, and EA is a length in the small triangle. We also have BD is a length in the big triangle minus a (corresponding) length in the small triangle, and EB is a length in the small triangle. This is why we had

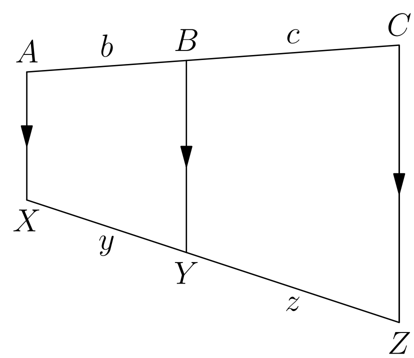
$$\frac{AC}{EA} = \frac{BD}{EB},$$

by the Part-to-Whole Principle.

But CD is a length in the big triangle, *not* a length in the big triangle minus a length in the small triangle. So $\frac{AC}{EA} \neq \frac{CD}{AB}$.

3 Parallel Lines Cut Proportions II: Where's the Triangle?

Problem 3: In the diagram, $AB = b$, $BC = c$, $XY = y$, and $YZ = z$. How are the numbers b, c, y, and z related? If $b = 6$, $c = 9$, and $y = 8$, find z.



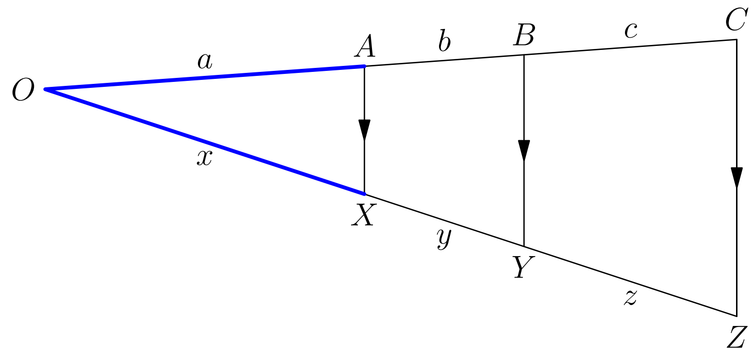
Hint: We have parallel lines like in the previous case... but we don't have a triangle! When we see a problem that looks like something we've seem before but doesn't quite match, we try to modify it in some way so that it *does* match something we've already seen.

Because Thales's Theorem looks similar, we see if we can *create* a triangle to apply it on. We can create a triangle in two ways. See if you can answer the following.

1. Can you *extend* line segments to make a triangle? (Extend segments that might intersect.)
2. Can you *move* line segments to make a triangle? (Try to make a triangle cut by $\overline{BY}$, and with one of the sides either equal to $\overline{AC}$ or $\overline{XZ}$.)

What information does Thales's Theorem give you when applied to this triangle?

Solution 1: To get a triangle, we can extend sides $\overline{AC}$ and $\overline{XZ}$ to meet at O. (What if they don't intersect? We'll talk about this at the end.) Then we get a triangle $\triangle OCZ$ cut by parallel lines, so are in the situation of Thales's Theorem.



In fact, since we have not one but *two* parallel lines cutting $\triangle OCZ$, we can apply AA similarity twice to get

$$\triangle OAX \sim \triangle OBY \sim \triangle OCZ.$$

From this, we have the proportions $OA : OB : OC = OX : OY : OZ$, or $a : (a+b) : (a+b+c) = x : (x+y) : (x+y+z)$. Note that $(a+b)-a = b$ and $(x+y)-x = y$. By the Part-to-Whole Principle we have $a : b = x : y$. In the same way, $(a+b+c)-(a+b) = c$ and $(x+y+z)-(x+y) = z$ give $a+b : c = x+y : z$. We get

$$a : b : c = x : y : z.$$

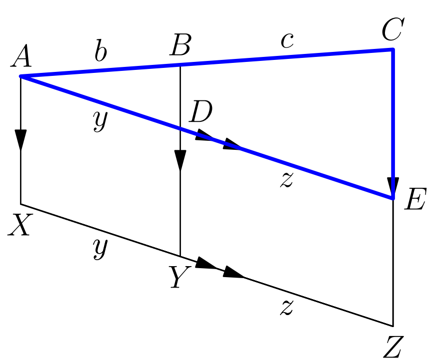
In particular,

$$\frac{b}{c} = \frac{y}{z}.$$

What about if $\overline{AC}$ doesn't intersect $\overline{XZ}$? In that case, $\overline{AC} \parallel \overline{XZ}$, so $ABYX$ and $YBCZ$ are parallelograms. Opposite sides of parallelograms are equal, so $b = y$ and $c = z$, and we also have $\frac{b}{c} = \frac{y}{z}$.

Solution 2: We'd like a triangle one of whose sides is $\overline{AC}$ (or $\overline{XZ}$), because the fact that there are parallel lines cutting $\overline{AC}$ and $\overline{XZ}$ suggest that we can use Thales's Theorem.

Draw a line through A parallel to $\overleftrightarrow{XZ}$ (which we can do by the parallel postulate) and let it intersect $\overline{BY}$ at D and $\overline{CZ}$ at E.



Now, opposites sides of $ADYX$ are parallel, so $ADYX$ is a parallelogram. The same is true of $DEZY$. Opposites sides in a parallelogram are congruent, so $AD = XY = y$ and $DE = YZ = z$.

Now we are in a position to apply Thales's Theorem! By parallel lines, $\triangle ABD \sim \triangle ACE$ so $\frac{AB}{BC} = \frac{AD}{DE}$, i.e.

$$\frac{b}{c} = \frac{y}{z}.$$

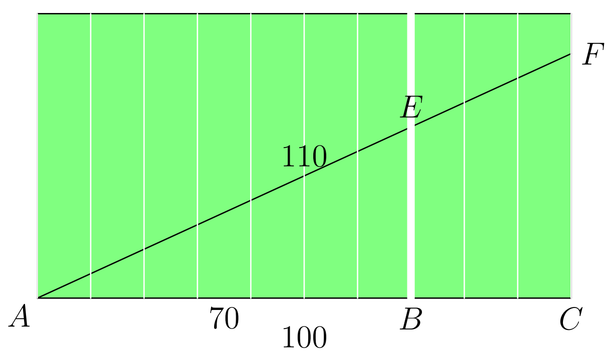
Both solutions teach us that creating triangles not already in the diagram is a powerful idea. We have multiple ways to create this triangle:

- in the first solution we created a triangle by *extending lines*, and
- in the second solution we created a triangle by *drawing a parallel line.*

> Sometimes the similar triangles will not already be in the diagram. Try to create triangles by extending lines or drawing parallel lines.

4 Problem Solving with Thales's Theorem

1. Connie and Eric are racing between the endlines of a 100-yard long football field. Because Eric thinks he is much better at running, his dad gives him a handicap: he picks a point on the other side that that is 110 yards away. Connie runs straight across the field, 100 yards.

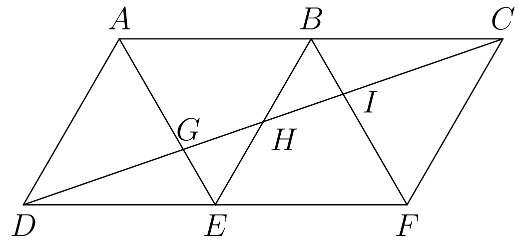


   It turned out, however, that Eric sorely misjudged his speed, and when Connie finishes he is still at the 70-yard line! (Connie is at C, Eric is at E, and Eric's finish point is F).

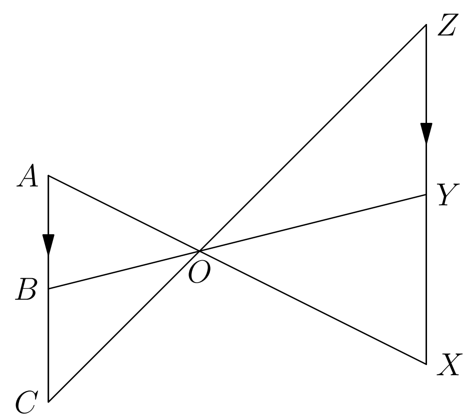
   (a) How far has Eric run?

   (b) How far does Eric have left to go?

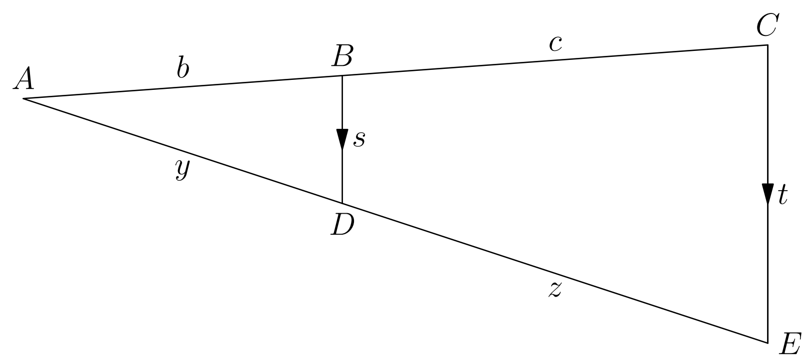
2. In the picture, $\triangle ADE$, $\triangle ABE$, $\triangle BEF$, and $\triangle BCF$ are all equilateral with side length 1. Suppose DC intersects AE, BE, and BF in G, H, and I, respectively. Find AG, BH, and BI.



3. Segments $\overline{AX}$ and $\overline{CZ}$ meet at O, and $\overline{AC} \parallel \overline{ZX}$. Let B and Y be points on $\overline{AC}$ and $\overline{XZ}$ such that B, O, and Y are on the same line.
   (a) Show that $\frac{AB}{BC} = \frac{XY}{YZ}$.
   (b) If $AB = 12$, $BC = 4$, and $ZY = 15$, find XY.



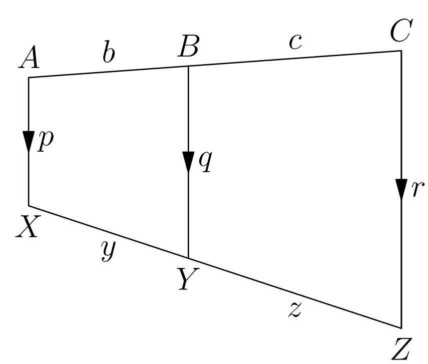
4. In the picture, $\overline{BD} \parallel \overline{CE}$, $AB = b < BC = c$, $AD = y$, $DE = z$, $BD = s$, and $CE = t$.



Prove that

$$\frac{b}{c} = \frac{y}{z} = \frac{s}{t-s}.$$

5. In the picture, $\overline{AX} \parallel \overline{BY} \parallel \overline{CZ}$, $p < q < r$, $AB = b$, $BC = c$, $XY = y$, and $YZ = z$.



(a) Prove that

$$\frac{b}{c} = \frac{y}{z} = \frac{q-p}{r-q}.$$

(b) Suppose that B is the midpoint of $\overline{AC}$ and Y is the midpoint of $\overline{XZ}$. Show that q is the average of p and r.

5 Lesson and Solutions

1. The lines on the football field are all parallel to each other, so by AA similarity, $\triangle EAB \sim \triangle FAC$. This means that

$$\frac{AB}{AC} = \frac{AE}{AF}.$$

   (a) Substituting numbers, we get

$$\frac{70}{100} = \frac{AE}{110} \qquad \Longrightarrow \qquad AE = \frac{7}{10} \cdot 110 = 77.$$

   Thus Eric has run $\boxed{77 \text{ feet}}$.

   (b) We have $EF = AF - AE = 110 - 77 = 33$, so Eric has $\boxed{33 \text{ feet}}$ left to run.

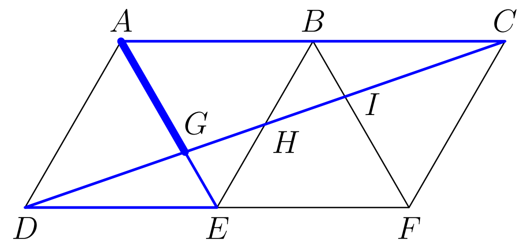
   Note: Parallel lines gives us equal angles and similar triangles. Similar triangles give proportions which allow us to calculate side lengths.

2. The angles in an equilateral triangle are equal to $60°$, so $\angle EAB = \angle AED = 60°$, and $\overline{AB} \parallel \overline{DE}$. To find the three lengths, we look at the triangles containing those lengths.

   Since $\overline{AC} \parallel \overline{DE}$, we get $\triangle GAC \sim \triangle GED$, and

$$\frac{GA}{GE} = \frac{AC}{ED} = 2.$$

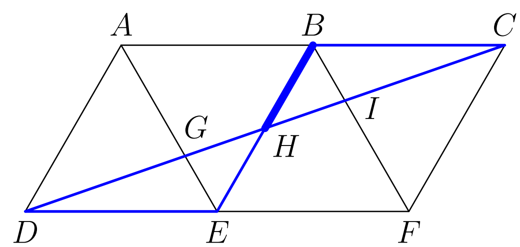
   Because $GA + GE = AE = 1$, we have $GA = \boxed{\frac{2}{3}}$.



   Since $\overline{BC} \parallel \overline{DE}$, we get $\triangle HBC \sim \triangle HED$, and

$$\frac{GA}{GE} = \frac{AC}{ED} = 1.$$

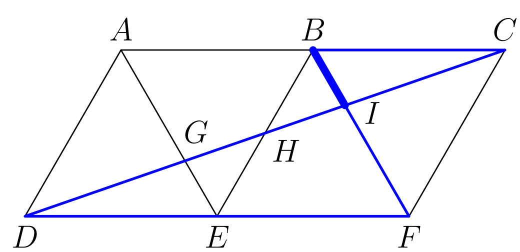
   (These triangles are actually congruent.) Because $BH + HE = BE = 1$, we have $BH = \boxed{\frac{1}{2}}$.



   In the same way, we get $\triangle IBC \sim \triangle IFD$, and

$$\frac{IB}{IF} = \frac{BC}{FD} = \frac{1}{2}.$$

   Because $BI + IF = 1$, we have $BI = \boxed{\frac{1}{3}}$. (Note this makes sense, because you turn the picture around $180°$, GA will get sent to IF and we already know $GA = \frac{2}{3}$.

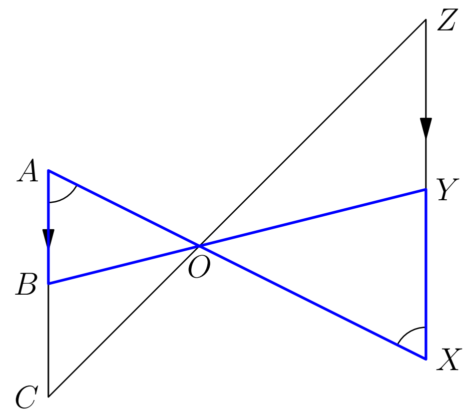


3. Unfortunately, none of the segments AB, BC, XY, and YZ are in the same triangle, so we cannot just apply similar triangles once and be done. But the diagram matches something we've seen several times. We know we can get similar triangles from parallel lines. We look for triangles involving the sides AB, BC, XY, and YZ.

(a)

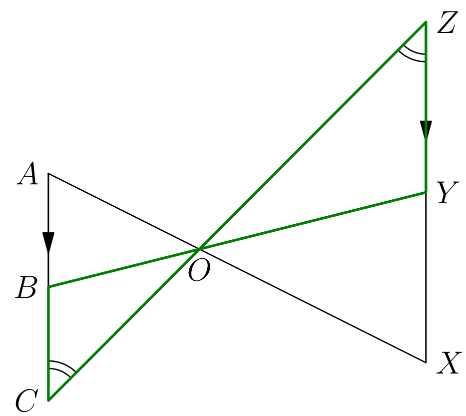
Because $\overline{AB} \parallel \overline{XY}$, we get $\angle OAB = \angle OXY$. Vertical angles $\angle AOB$ and $\angle XOY$ are equal. This means $\triangle AOB \sim \triangle XOY$ by AA. We get the proportion

$$\frac{}{AB}BO = \frac{XY}{YO}. \tag{1}$$



Because $\overline{BC} \parallel \overline{YZ}$, we get $\angle OCB = \angle OZY$. Vertical angles $\angle COB$ and $\angle ZOY$ are equal. This means $\triangle COB \sim \triangle ZOY$ by AA. We get the proportion

$$\frac{}{BC}BO = \frac{YZ}{YO}. \tag{2}$$

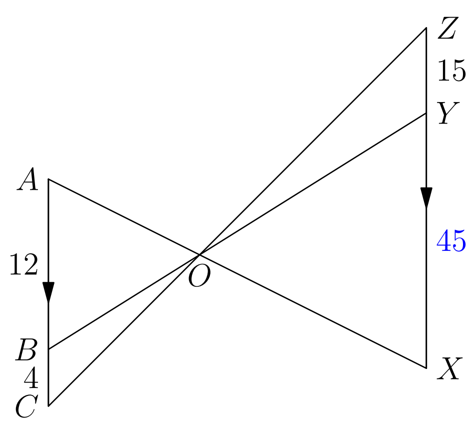


To relate AB to BC we divide (1) by (2) to get

$$\frac{AB}{BC} = \frac{XY}{YZ}.$$

(b) We substitute in the values $AB = 12$, $BC = 4$, $ZY = 15$ (in the correct slots!) to get

$$\frac{12}{4} = \frac{XY}{15} \implies XY = \boxed{45}.$$



You may have to use more than one pair of similar triangles, and multiply ratios you get from different pairs of similar triangles.

To relate AB and BC, we used the fact that OB was in a triangle with AB, and OB was in another triangle with BC.

4. We give 2 solutions.

Solution 1: We do the same thing that we did in the proof of Thales's Theorem, this time keeping in mind BD and CE. By parallel lines, $\triangle ABD \sim \triangle ACE$ and we the ratio of the sides of $\triangle ACE$ to those of $\triangle ABD$ is

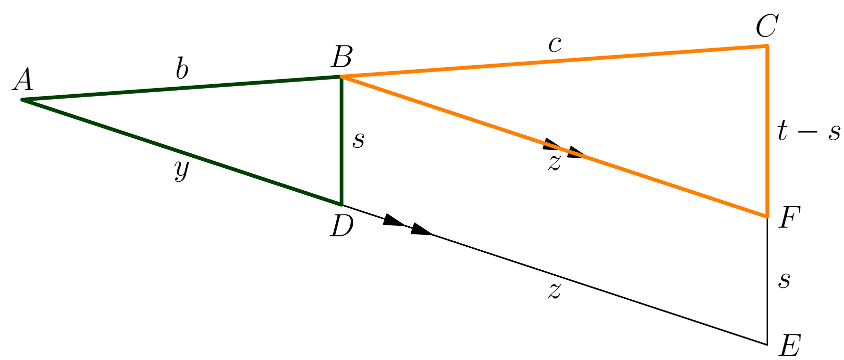
$$\frac{b+c}{b} = \frac{y+z}{y} = \frac{t}{s}.$$

Subtracting 1 gives

$$\frac{c}{b} = \frac{z}{y} = \frac{t-s}{s}.$$

Taking the reciprocals gives $\frac{b}{c} = \frac{y}{z} = \frac{s}{t-s}$.

Solution 2: Let's draw in line parallel to $\overline{AE}$ passing through B, and have it intersect $\overline{CE}$ at F.



Then $BDEF$ is a parallelogram, so opposite sides are equal: $BF = DE = z$ and $FE = BD = s$. By parallel lines $\overline{BF} \parallel \overline{AE}$ and AA similarity, $\triangle CBF \sim \triangle CAE$. By parallel lines $\overline{BD}$ and $\overline{CE}$ and AA similarity, $\triangle CAE \sim \triangle BAD$. This means

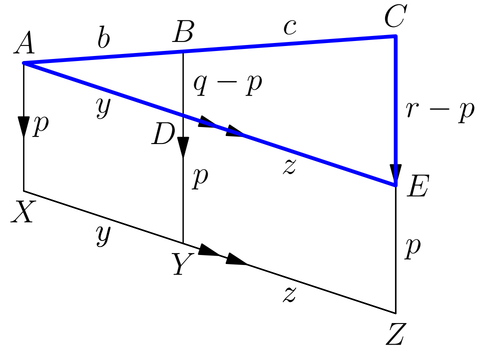
$$\triangle CBF \sim \triangle BAD,$$

and we get

$$\frac{AB}{BC} = \frac{AD}{BF} = \frac{BD}{CF} \qquad \Longrightarrow \qquad \frac{b}{c} = \frac{y}{z} = \frac{s}{t-s}.$$

5. We've seen this configuration before, just haven't dealt with the lengths AX, BY, and CZ.

   (a) We draw a line parallel to $\overline{XZ}$ intersecting $\overline{BY}$ at D and $\overline{CZ}$ at E.



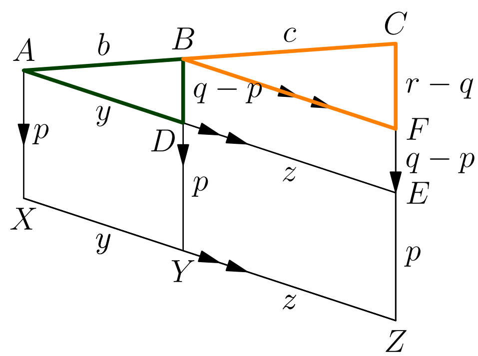
Now $ADYX$ and $DEZY$ are parallelograms, so their opposite sides are equal:

$$\begin{aligned} AD &= XY = b \\ DE &= YZ = z \\ CZ &= DY = AX = p. \end{aligned}$$

Now we can apply the previous problem to get

$$\frac{b}{c} = \frac{y}{z} = \frac{(q-p)}{(r-p)-(q-p)} = \frac{q-p}{r-q}.$$

If we put the two pictures together we get this nice picture.



(b) If B is the midpoint of $\overline{AC}$, then $b = c$. Thus

$$1 = \frac{q-p}{r-p} \implies q - p = r - p \implies q = \frac{p+r}{2}.$$

Note: In this problem we once again start with a diagram that doesn't have triangles, and then create similar triangles by adding in a parallel segment.