

Lecture 18 — Generating functions

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1 Introduction

Generating functions offer a way to convert combinatorics questions into purely algebraic and analytic questions. The basic idea is this: we take a sequence a_0, a_1, a_2, \dots , usually representing something of combinatorial interest—for example, a_n could be the number of ways to make change with n cents (see the first lecture)—and then associate with it a power series.

Definition 1.1: The **generating function** of the sequence a_0, a_1, a_2, \dots is the *formal series*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

To find the a_n , or identities involving the a_n , we rewrite the generating function in terms of familiar functions using algebra and calculus, manipulate the equation, and then extract the coefficients.

In Sections 2 we develop the algebraic and analytic machinery of generating functions; in Sections 3 and 4 we give applications to algebra and combinatorics. An impatient reader may also wish to skip to the examples, and look up the relevant algebraic and analytic facts as needed.

In our above definition, we said that a generating function is a *formal series*. But what exactly is a formal series?

2 Formal series

2.1 The ring of formal series

Definition 2.1: Let R be a ring. The **ring of formal (power) series** in R is the set of all expressions

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

The ring of formal series will be denoted by $R[[x]]$ (compare to the ring of polynomials $R[x]$).

We will work exclusively with $R = \mathbb{R}$, i.e. formal series with real coefficients. Just like polynomials, we can repeat this process to obtain formal series in more than one variable, to get $R[[x]][[y]] = R[[x, y]]$, etc.

Note a formal series is *not* a function. Right now we don't care whether the sum $\sum_{n=0}^{\infty} a_n x^n$ "converges." Don't think of x as being the input value for a function; think of it just as the symbol x . Note this perspective was useful in our development of polynomials; it will also be essential for our development of formal series. (In Section 2.4 we *will* look at formal series as functions.)

We said R is a ring, meaning that it has addition and multiplication defined, satisfying the "usual" properties (see lecture 8). If f and g are power series, we define $f + g$ by adding corresponding terms and multiplication by multiplying every term in f and every term in g , and adding. In symbols, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$f(x)g(x) = \sum_{n=0}^{\infty} (a_n b_0 + a_{n-1} b_1 + \cdots + a_1 b_{n-1} + a_0 b_n) x^n. \quad (1)$$

Note that the coefficient of fg is the sum of products of coefficients in f and coefficients in g , where the corresponding exponents sum to n .¹ This generalizes to the product of m factors in the obvious way (for 3 sequences $f(x)g(x)h(x) = \sum_{i+j+k=n} a_i b_j c_k x^n$, and so on), and will be important for combinatorial interpretations.

It is simple to check that we have additive inverses and that the normal rules distributivity, commutativity, and associativity hold. We will leave this for the fastidious reader.

There is one more thing we could do: we can compose power series, that is, substitute one in for the other.

Definition 2.2: Given $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=1}^{\infty} b_n x^n$, define the **composition** $f \circ g$ to be

$$f \circ g(x) = \sum_{n=0}^{\infty} a_n g(x)^n.$$

Note this makes sense, because for $n > 0$, the coefficient of x^n in $f \circ g$ will only depend on the partial sum $\sum_{m=0}^n a_m g(x)^m$ (why?). Note that we require the constant term of g to be 0 to avoid an infinite sum for the constant term. This way, the constant term of $f \circ g$ equals a_0 , the constant term of f .

2.2 Inverses

If f is a power series, a logical question is whether $\frac{1}{f}$ or f^{-1} is well-defined. You may know the geometric series formula,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

¹The sequence of coefficients of $f(x)g(x)$ is called the *convolution* of $\{a_n\}$ and $\{b_n\}$.

which says that $1 + x + x^2 + \dots$ is the multiplicative inverse of $\frac{1}{1-x}$ in $\mathbb{R}[[x]]$. Indeed you can check using our definition of multiplication that $(1-x)(1+x+x^2+\dots) = 1$. Now we see under what conditions $\frac{1}{f}$ and f^{-1} exist.

Theorem 2.3: Suppose $f = \sum_{n=0}^{\infty} a_n x^n$ where $a_0 \neq 0$. Then there exists a unique g such that $fg = 1$. g is called the **multiplicative inverse** of f and denoted by $\frac{1}{f}$.

Moreover, the coefficient of x^n in g only depends on a_0, \dots, a_n .

Proof. We find the coefficients b_0, b_1, \dots of g inductively. To make the constant term 1 we need $b_0 = \frac{1}{a_0}$.

Suppose we know that g starts with $b_0 + b_1x + \dots + b_kx^k$ and $f(x)(b_0 + b_1x + \dots + b_kx^k)$ has coefficients of $1, x, \dots, x^k$ all equal to 0. Now we define b_{k+1} so that the coefficients of $1, x, \dots, x^{k+1}$ in $f(x)(b_0 + b_1x + \dots + b_kx^k + b_{k+1}x^{k+1})$ are equal to 0. The coefficients of $1, x, \dots, x^k$ are not affected by the addition of $b_{k+1}x^{k+1}$, so we just need the coefficient of x^{k+1} to be 0:

$$a_0b_{k+1} + a_1b_k + \dots + a_{k+1}b_0 = 0.$$

So let

$$b_{k+1} = \frac{-a_1b_k - \dots - a_{k+1}b_0}{a_0},$$

and this is the only value that works.

Then $g = \sum_{n=0}^{\infty} b_n x^n$. □

Theorem 2.4: Suppose $f = \sum_{n=1}^{\infty} a_n x^n$. Then there exists a unique g such that $f \circ g = 1$. Moreover, $g \circ f$ as well. g is called the **inverse** of f and denoted by f^{-1} .

Moreover, the coefficient of x^n in g depends only on a_1, \dots, a_n .

Proof. Exercise. (Similar to the above proof.) □

2.3 Derivatives

We define the derivative the same way as in calculus.

Definition 2.5: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. The **derivative** of f is

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

The following can be proved just like it is proved for polynomials. (Alternatively, conclude it from the corresponding result for polynomials, by noting that the coefficients of x^n depends on only finitely many terms in the expressions below.)

Proposition 2.6: Let f, g be two power series and c be a constant. Then

1. (Linearity) $(cf + g)' = cf' + g'$.
2. (Product rule) $(fg)' = f'g + fg'$.
3. (Chain rule) For g with constant term 0, $(f \circ g)' = (f' \circ g)g'$.

We can state the product rule in a slightly different form, that will often be of use.

Proposition 2.7 (Logarithmic differentiation): Suppose $f = f_1 \cdots f_n$ where f_1, \dots, f_n are power series. Then

$$f' = f \left(\frac{f_1'}{f_1} + \cdots + \frac{f_n'}{f_n} \right).$$

Proof. By the product rule,

$$f' = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n'.$$

which gives the above after using $f = f_1 \cdots f_n$.

Alternatively (and this is why the proposition is called logarithmic differentiation) take \ln of both sides, differentiate, and use the chain rule, and note that $(\ln x)' = \frac{1}{x}$:

$$\begin{aligned} \ln f &= \ln f_1 + \ln f_2 + \cdots + \ln f_n \\ \frac{f'}{f} &= \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_n'}{f_n} \end{aligned} \tag{2}$$

which rearranges into the same identity.

Why is this justified? This identity is true if the f_i 's are all polynomials, by calculus and Theorem 2.8 in the next subsection. Moreover, the coefficient of x^n on both sides of (2) will stay the same if each f, f_i is replaced by a polynomial with coefficients of $1, x, \dots, x^{n+1}$ agreeing (so the coefficients of $1, \dots, x^n$ of f_1', \dots, f_n' will be the same; use the last statement in Theorem 2.3), so the conclusion follows from the finite case. \square

2.4 Analytic point of view

From calculus, we know how to expand functions in power series $\sum_{n=0}^{\infty} a_n x^n$. For each power series there is a r such that it converges absolutely² for $|x| < r$, but not for $|x| > r$. In this domain we may differentiate and integrate and still have a convergent power series.

A table of power series and generating functions can be found in the Appendix. For example,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots .$$

We have equality in the *calculus* sense: both sides are equal as functions, i.e. give the same result when we plug in any value for x .

An identity in formal power series automatically transfers to an identity in functions, provided the functions converge. The following gives the opposite implication.

Theorem 2.8: Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are two absolutely convergent power series on an interval $(-\varepsilon, \varepsilon)$ around 0, and $f(x) = g(x)$ for $x \in (-\varepsilon, \varepsilon)$. Then $a_n = b_n$ for all n .

Proof. Let $h(x) = f(x) - g(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = a_n - b_n$. If c_n is not the zero sequence, let m be the least integer so that $c_m \neq 0$. Then $h(x) = x^m(c_m + j(x))$ where $j(x) = c_{m+1}x + c_{m+2}x^2 + \cdots$. Since power series are continuous, $j(x)$ is continuous. Note it equals 0 at $x = 0$, so is close to 0 for x close to 0. Taking $x \neq 0$ sufficiently small, we have $|j(x)| < |a_m|$, so $a_m + j(x) \neq 0$ and $h(x) \neq 0$, contradiction. \square

²That is, $\sum_{n=0}^{\infty} |a_n x^n|$ is finite.

As a simple example, we know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

holds for $x \in (-1, 1)$, so this is an identity of power series. (Recall we could also conclude this by directly multiplying $(1-x)$ and $1 + x + x^2 + \dots$.)

Why is calculus useful?

1. It allows us to turn power series into familiar functions, which we can easily multiply, divide, and differentiate, taking full advantage of algebraic identities (ex. $\sin^2 x + \cos^2 x = 1$ —it would be annoying to the corresponding power series satisfy this identity using the definition of multiplication for formal series!).
2. It allows us to substitute in values for x to get identities involving the coefficients, or take limits (see Section 4.4).
3. We could use the tools of complex analysis to analyze the growth of the sequence. (This is beyond the scope of this lecture.)

3 Applications to Algebra

We give two applications of generating functions to algebra. First, generating functions are useful for finding closed formulas for sequences. The general strategy is the following.

1. Convert the recurrence relation for a_n into an identity involving the generating function f .
 - (a) Shift the sequence by multiplying by powers of x . For fixed i , the generating function for a_{n-i} is $f(x)x^i$; the generating function of a_{n+i} is $x^{-i}(f(x) - a_0 - \dots - a_{i-1}x^{i-1})$.
 - (b) The sequence $(n+1)a_{n+1}$ is obtained by taking the derivative of f .
 - (c) A term like $a_n b_0 + \dots + a_0 b_n$ is obtained by multiplying power series.
2. If we want a closed formula for the sequence, solve for f and extract the coefficients.

If we want to prove an identity or simplify an expression involving the sequence, massage the identity for f so that the desired identity can be read off from equating coefficients of x^n .

3.1 Solving recurrences

Theorem 3.1: Suppose that $c_0 \neq 0$ and the roots of $P(x) := x^k + c_{k-1}x^{k-1} + \dots + c_1x + c_0 = 0$ are distinct and equal to r_1, \dots, r_k . The general solution to the linear recurrence

$$a_n + c_{k-1}a_{n-1} + \dots + c_0a_{n-k} = 0, \quad n \geq k$$

where $c_0 \neq 0$, is

$$a_n = b_1 r_1^n + b_2 r_2^n + \dots + b_k r_k^n. \quad (3)$$

Proof. Let $f(x)$ be the generating function for a_n , so that $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We multiply $f(x)$ by powers of x to get the shifted sequences $\{a_{n-i}\}$:

$$x^i f(x) = \sum_{n=0}^{\infty} a_n x^{n+i} = \sum_{n=i}^{\infty} a_{n-i} x^n.$$

Thus the recurrence relation becomes

$$f(x) + c_{k-1}x f(x) + \cdots + c_0 x^k f(x) = 0,$$

because the coefficient of x^n on left hand side is $a_n + c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k}$, which is equal to 0. Dividing by $1 + c_{k-1}x + \cdots + c_0 x^k$, we obtain $f(x) = 0$. Wait, that's not right. What went wrong?

The identity $a_n + c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k} = 0$ is only valid for $n \geq k$. (Moral: always be careful with small cases.) So for $n < k$, there is no restriction on the coefficient of x^n on the LHS. So instead, we should write

$$f(x) + c_{k-1}x f(x) + \cdots + c_0 x^k f(x) = d_0 + d_1 x + \cdots + d_{k-1} x^{k-1} \tag{4}$$

for some b_0, \dots, b_{k-1} . Now we use the following:

Theorem 3.2 (Partial fraction decomposition): Suppose $p(x)$ and $q(x)$ are polynomials with $\deg p < \deg q = k$, and q has distinct roots r_1, \dots, r_n . Then there are b_1, \dots, b_k such that

$$\frac{p(x)}{q(x)} = \frac{b_1}{x - r_1} + \cdots + \frac{b_k}{x - r_k}.$$

Hence dividing (4) by $1 + c_{k-1}x + \cdots + c_0 x^k$, noting that $\frac{1}{r_i}$ are the zeros of $1 + c_{k-1}x + \cdots + c_0 x^k = x^n P\left(\frac{1}{x^n}\right)$, and using the above theorem gives

$$\begin{aligned} f(x) &= \frac{b_1}{x - \frac{1}{r_1}} + \cdots + \frac{b_k}{x - \frac{1}{r_k}} \\ &= \frac{-r_1 b_1}{1 - r_1 x} + \cdots + \frac{-r_n b_n}{1 - r_n x}. \end{aligned}$$

for some b_i . (Conversely, given $f(x)$ in the above form, (4) holds for some b_i , so $f(x)$ gives a valid sequence.) Now using the geometric series formula gives

$$f(x) = \sum_{i=1}^k -r_i b_i (1 + r_i x + r_i^2 x^2 + r_i^3 x^3 + \cdots).$$

Extracting the n th coefficient gives

$$a_n = \sum_{i=1}^k -r_i b_i r_i^n = \sum_{i=1}^k b'_i r_i^n$$

after setting $b'_i = -r_i b_i$. □

Remark 3.3: In the general case, the general solution to the linear recurrence

$$a_n + c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k} = 0, \quad n \geq k$$

where $c_0 \neq 0$, is given by

$$a_n = \sum_r P_r(x)r^n$$

where the sum is over the roots r of the characteristic polynomial $x^k + c_{k-1}x^{k-1} + \dots + c_0$ and $P_r(x)$ is a polynomial of degree less than the multiplicity of the root r in P . The above proof can be adapted to solve the general version.

Note that generating sequences do not necessarily make for the most elegant or short proofs. If we know the general form of the solution, we can prove show that it works just by substituting in, and find an argument to show directly that all solutions are of that form. But generating sequences are especially useful if we do not know the solution, since they offer a methodical way to find it. For some deep results, though, no proof without generating functions is known!

3.2 Newton sums

Let x_1, x_2, \dots be variables. In this subsection we develop the Newton formulas relating the symmetric polynomials in the x_i

$$s_n = \sum_{1 \leq i_1 < \dots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}$$

and the power sums

$$p_n = \sum_{i \geq 1} x_i^n.$$

(For example, when we have 3 variables, $s_2 = x_1x_2 + x_1x_3 + x_2x_3$ and $p_2 = x_1^2 + x_2^2 + x_3^2$. For convenience of notation we don't restrict the number of variables; if we are working with k variables we could just set $0 = x_{k+1} = x_{k+2} = \dots$. By convention $s_0 = 1$.)

We note that the s_i and p_i are the coefficients of powers of t of the following generating functions.

$$S(t) = \prod_{i \geq 1} (1 + x_i t) = 1 + s_1 t + s_2 t^2 + \dots$$

$$P(t) = \sum_{i \geq 1} \frac{x_i}{1 - x_i t} = \sum_{i \geq 1} (x_i + x_i^2 t + \dots) = p_1 + p_2 t + \dots$$

Why do these expansions hold? Each term in the expansion of $S(t)$ is a term of the form $x_{i_1} \dots x_{i_n} t^n$ where the x_{i_j} are distinct, since we can only get a factor of x_{i_j} from the factor $(1 + x_{i_j} t)$. Since each x_i always comes with a factor of t , t acts as a "counter" giving the total number of variables. Grouping the terms with t^n we get all combinations of products of n of the x_i .

For $P(t)$ the argument is simpler: expand in geometric series as shown above to get that the coefficients of t^{n-1} are the n th powers of the x_i .

We want an identity involving s_i and p_i so we look for an identity involving $S(t)$ and $P(t)$. First we turn the $1 + x_i t$ into $1 - x_i t$: define

$$\Psi(t) = S(-t) = \prod_{i \geq 1} (1 - x_i t) = 1 - s_1 t + s_2 t^2 - s_3 t^3 + \dots$$

Now use logarithmic differentiation (Proposition 2.7) to get

$$\frac{\Psi'(t)}{\Psi(t)} = - \sum_{i \geq 1} \frac{x_i}{1 - x_i t}.$$

The right-hand side is just $-P(t)$ (TA-DA!). Thus we get

$$-\Psi'(t) = \Psi(t)P(t).$$

Now we match coefficients of t^{n-1} on both sides. Note

$$-\Psi'(t) = s_1 - 2s_2t + 3s_3t^2 - \dots$$

so the coefficient of t^{n-1} is $(-1)^{n+1}ns_n$. To get the coefficients of the right-hand side, note that a term containing t^{n-1} on the RHS comes from multiplying $(-1)^i s_i t^i$ in $\Psi(t)$ and a term $p_{n-i}t^{n-i-1}$ in $P(t)$. Thus the coefficients of t^{n-1} are

$$(-1)^{n+1}ns_n = \sum_{i=0}^{n-1} (-1)^i s_i p_{n-i} = s_0 p_n - s_1 p_{n-1} + \dots + (-1)^{n-1} s_{n-1} p_1$$

which is the Newton sum formula.

4 Applications to Combinatorics

In some combinatorial problems, we find a recurrence relation for the sequence of interest, and then turn it into a generating series (see Section 4.2). In others, we build the generating function directly by multiplying together several power series, and note that every resulting term is obtained by choosing one term from each of the series; the exponents of the variable acts should like some “weight,” or some value of interest to the problem. (See Theorem 4.4 and Example 4.9.)

See also the first lecture (<http://onlinemathcircle.com/wp-content/uploads/2010/12/1-beauty.pdf>) for an application of generating functions to counting change.

4.1 Binomial coefficients

By the binomial formula,

$$(1 + x)^n = \sum_{i=0}^n \binom{n}{i} x^i.$$

Thus $(1 + x)^n$ is the generating function for the sequence $\binom{n}{i}, i \geq 0$. This is a particularly simple function; in fact it is just a polynomial! We can use this and our knowledge of multiplying generating functions (or just polynomials) to get a quick identity.

Example 4.1 (Vandermonde convolution):

$$\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}.$$

Proof. Consider the identity

$$(1 + x)^m (1 + x)^n = (1 + x)^{m+n}.$$

The i th coefficient of $(1 + x)^m$ is $\binom{m}{i}$, the $(k - i)$ th coefficient of $(1 + x)^n$ is $\binom{n}{k-i}$, and the k th coefficient of $(1 + x)^{m+n}$ is $\binom{m+n}{k}$. The identity follows for the formula for the product of generating functions, (1). □

4.2 Catalan numbers

The Catalan numbers are a famous sequence in combinatorics, and appear in many different contexts. Below we give several equivalent definitions.

Definition 4.2: The Catalan number C_n is the

1. Number of binary trees with n vertices. A binary tree is a rooted tree so that each vertex has at most two outgoing edges going to the left or right. For example, there are 5 rooted trees with 3 vertices:



2. Number of sequences of n X 's and n Y 's such that in any initial segment, the number of X 's is at least the number of Y 's.
3. Number of paths from $(0, 0)$ to (n, n) , that go right or up one unit at each step and that do not go above the main diagonal $y = x$.
4. Number of ways to triangulate a $(n + 2)$ -gon.

We first derive an identity for the C_n . We claim

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k} = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0. \tag{5}$$

using any of the definitions.

To count the number of binary trees with $n + 1$ vertices, note that if we take off the root vertex of a tree with $n + 1$ vertices, the tree is split into two smaller binary trees, of sizes k and $n - k$, for some $0 \leq k \leq n$. There are $C_k C_{n-k}$ possibilities for these smaller trees. Moreover, this operation is reversible. Thus summing over $0 \leq k \leq n$ gives (5).

To see the formula holds for the second definition, given a sequence of X 's and Y 's satisfying the conditions (a *ballot sequence*), let k be the smallest nonnegative integer so that there are as many X 's as Y 's in the first $2(k + 1)$ entries. Now delete the first entry and the $2(k + 1)$ th entry, to get two sequences of lengths $2k$ and $2(n - k)$. The first is a ballot sequence by minimality of k and the second is a ballot sequence since there are as many X 's as Y 's before the second sequence. The construction is reversible, so we get the same formula as before.

The third definition is equivalent to the second because we can associate taking a step to the right with X and taking a step up as Y . The last definition is left to the reader.

Now let y be the generating function for C_n . We turn our recurrence relation into an identity involving f . C_{n+1} is the coefficient of x^n in $\frac{y-1}{x}$ —note we subtract out the constant term before dividing by x , so that the result is a power series. The RHS of (5),

$\sum_{k=0}^n C_k C_{n-k}$, is the coefficient of x^n in y^2 . Thus using the quadratic formula we get

$$\begin{aligned} \frac{y-1}{x} &= y^2 \\ xy^2 - y + 1 &= 0 \\ y &= \frac{1 \pm \sqrt{1-4x}}{2x}. \end{aligned} \tag{6}$$

Now using the binomial formula $((1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$ where $\binom{r}{n} = \frac{r^n}{n!} = \frac{r(r-1)\dots(r-n+1)}{n!}$ we get

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n = 1 - \sum_{n=1}^{\infty} \frac{2}{n} \binom{2(n-1)}{n-1} x^n$$

since for $n > 0$,

$$\begin{aligned} \binom{\frac{1}{2}}{n} (-4)^n &= \frac{1}{n!} \binom{1}{2} \binom{-1}{2} \dots \binom{-2n+3}{2} (-4)^n = -\frac{1}{n!} 2^n (2n-3)(2n-5)\dots 1 \\ &= -\frac{1}{n!} 2^n \frac{(2n-2)!}{(2n-2)(2n-4)\dots 2} = -\frac{2(2(n-1))!}{n!(n-1)!} = -\frac{2}{n} \binom{2(n-1)}{n-1} \end{aligned}$$

In (6) we take the $-$ sign so that the constant term of the numerator 0 and y is a valid power series. Then

$$y = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2(n-1)}{n-1} x^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

Hence $C_n = \frac{1}{n+1} \binom{2n}{n}$.

4.3 Partitions

One of the most famous generating functions in combinatorics is that of the partition function.

Definition 4.3: Let m be a nonnegative integer. A **partition** of m is a way to write it as the sum of positive integers, with order not taken into account. In other words, it is a sequences of positive integers $(\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \dots \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = m$.

Define $p(n)$ to be the number of partitions of n . By convention, $p(0) = 1$.

Theorem 4.4: The generating function of the sequence $p(n)$ is

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n}. \tag{7}$$

Proof. A partition of n can be specified by a sequence (a_1, a_2, \dots) where a_i is the number of times i appears in the partition. In order for this to specify a partition of n , we must have

$$\sum_{i=1}^n i a_i = n.$$

The number of such sequences a_i is simply the coefficient of x^n in

$$\prod_{i=1}^{\infty} (1 + x^i + x^{2i} + x^{3i} + \dots)$$

since each term x^n in the expansion comes from multiplying together terms of the form x^{ia_i} , where x^{ia_i} is taken from the i th factor in the product above. Using the geometric series formula gives (7). \square

We use the generating function to give an identity for $p(n)$.

Example 4.5: For a positive integer m let $\sigma(m)$ denote the sum of divisors of m . Then

$$n \cdot p(n) = \sum_{i=1}^n \sigma(i)p(n-i).$$

Proof. Note $n \cdot p(n)$ is the coefficient of x^{n-1} in $F'(x)$:

$$F'(x) = \sum_{n=1}^{\infty} np(n)x^{n-1}.$$

Now letting $f_n(x) = \frac{1}{1-x^n}$, we have $F(x) = \prod_{n=1}^{\infty} f_n(x)$. Note

$$\frac{f'_n}{f_n} = \frac{nx^{n-1}}{(1-x^n)^2}(1-x^n) = \frac{nx^{n-1}}{1-x^n}$$

so by logarithmic differentiation³

$$F'(x) = F(x) \sum_{m=1}^{\infty} \frac{f'_m}{f_m} = F(x) \sum_{m=1}^{\infty} \frac{mx^{m-1}}{1-x^m}. \tag{8}$$

Claim 4.6:

$$\sum_{m=1}^{\infty} \frac{mx^{m-1}}{1-x^m} = \sum_{t=1}^{\infty} \sigma(t)x^{t-1}.$$

Proof. We expand using the geometric series formula and then exchange the order of summation.

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{mx^{m-1}}{1-x^m} &= \sum_{m=1}^{\infty} mx^{m-1} \sum_{k=0}^{\infty} x^{mk} \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} mx^{mk-1} \\ &= \sum_{m>0, m|t} mx^{t-1} \\ &= \sum_{t=1}^{\infty} \left(\sum_{m|t} m \right) x^{t-1} \\ &= \sum_{t=1}^{\infty} \sigma(t)x^{t-1}. \end{aligned}$$

³Note there are infinitely many factors in the product. This is okay, because the coefficient of x^n will be the same in $F'(x)$ as in $(f_1 \cdots f_{n+1})'$, so the equation follows from the finite version for each n .

What we did above, with less math notation, is just switch from reading the rows to reading the columns of the below sum:

$$\begin{array}{rcccccccc}
 \frac{1}{1-x} & = & 1+ & x+ & x^2+ & x^3+ & x^4+ & x^5+ & \dots \\
 \frac{2x}{1-x^2} & = & & 2x+ & & 2x^3+ & & 2x^5+ & \dots \\
 \frac{3x^2}{1-x^3} & = & & & 3x^2+ & & & 3x^5+ & \dots \\
 \frac{4x^3}{1-x^4} & = & & & & 4x^3+ & & & \dots \\
 \frac{5x^4}{1-x^5} & = & & & & & 5x^4+ & & \dots \\
 \frac{6x^5}{1-x^6} & = & & & & & & 6x^5+ & \dots \\
 \vdots & & & & & & & & \vdots \\
 \hline
 & & \sigma(1)+ & \sigma(2)x+ & \sigma(3)x^2+ & \sigma(4)x^3+ & \sigma(5)x^4+ & \sigma(6)x^5+ & \dots
 \end{array}$$

□

Putting the above result into (8) we get

$$\sum_{n=1}^{\infty} np(n)x^{n-1} = F'(x) = F(x) \sum_{t=1}^{\infty} \sigma(t)x^{t-1} = \left(\sum_{n=0}^{\infty} p(n)x^n \right) \left(\sum_{t=1}^{\infty} \sigma(t)x^{t-1} \right).$$

Matching the coefficients of x^{n-1} on both sides gives the desired result. □

The generating function (7) for $p(n)$ can in fact be used to obtain much more information about $p(n)$, such as congruences like $p(5n + 4) \equiv 0 \pmod{5}$, and information on its asymptotic growth.

4.4 Arithmetic progressions

For some problems we may use apply calculus to get information about the generating functions, such as taking a limit.

Example 4.7: Suppose the the set of natural numbers is partitioned into arithmetic sequences with first terms a_1, \dots, a_k and common differences r_1, \dots, r_k , respectively. Then

1. $\frac{1}{r_1} + \dots + \frac{1}{r_k} = 1$.
2. $\frac{a_1}{r_1} + \dots + \frac{a_k}{r_k} = \frac{k-1}{2}$.

Proof. Every natural number occurs in exactly one of the sequences $\{a_1+nr_1\}_{n=0}^{\infty}, \dots, \{a_k+nr_k\}_{n=0}^{\infty}$, so

$$\sum_{n=0}^{\infty} x^{a_1+nr_1} + \dots + \sum_{n=0}^{\infty} x^{a_k+nr_k} = \sum_{n=0}^{\infty} x^n.$$

Using the geometric series formula, this becomes

$$\frac{x^{a_1}}{1-x^{r_1}} + \dots + \frac{x^{a_k}}{1-x^{r_k}} = \frac{1}{1-x}.$$

This identity holds for $|x| < 1$. Multiply through by $1-x$:

$$\frac{x^{a_1}}{x^{r_1-1} + \dots + x + 1} + \dots + \frac{x^{a_k}}{x^{r_k-1} + \dots + x + 1} = 1, \quad |x| < 1 \tag{9}$$

Now take the limit as $x \rightarrow 1^-$ to get

$$\frac{1}{r_1} + \dots + \frac{1}{r_k} = 1. \tag{10}$$

as needed.

To get an identity involving the a_i , we differentiate (9) to bring the a_i “out front”:

$$\sum_{i=1}^k \frac{a_i x^{a_i} (x^{r_i-1} + \dots + 1) - x^{a_i} ((r_i - 1)x^{r_i-2} + \dots + 1)}{(x^{r_i-1} + \dots + x + 1)^2} = 0, \quad |x| < 1$$

Now take the limit as $x \rightarrow 1^-$:

$$\sum_{i=1}^k \frac{a_i r_i - \frac{r_i(r_i-1)}{2}}{r_i^2} = -\frac{k}{2} + \sum_{i=1}^{\infty} \left(\frac{a_i}{r_i} + \frac{1}{2r_i} \right) = 0.$$

Now use (10) to get

$$\sum_{i=1}^k \frac{a_i}{r_i} = \frac{k-1}{2}.$$

□

4.5 Extracting coefficients

In this section we consider generating functions which are polynomials. Sometimes, rather than finding a specific coefficient, we want to find a sum of coefficients, for instance the sum of coefficients of x^n with n divisible by k . To do this, we evaluate the polynomial at the k th roots of unity and take the average.⁴

Lemma 4.8: Let $\omega = e^{\frac{2\pi i}{k}}$. Then the sum of the coefficients of x^n with $k|n$ in the polynomial $P(x)$ is

$$\frac{P(1) + P(\omega) + \dots + P(\omega^{k-1})}{k}.$$

Proof. For a k th root of unity ζ ,

$$1 + \zeta + \dots + \zeta^{k-1} = \begin{cases} 0, & \zeta \neq 1 \\ k, & \zeta = 1. \end{cases}$$

If $P(x) = \sum_{n=0}^{\infty} a_n x^n$, the the contribution to the sum $P(1) + P(\omega) + \dots + P(\omega^{k-1})$ from the x^n terms is $a_n(1 + \omega^n + \dots + \omega^{n(k-1)})$, which by the above, is $a_n k$ when $\omega^n = 1$ i.e. $k|n$, and 0 otherwise. □

(Query: Given r , how would you find the sum of coefficients of x^n with $n \equiv r \pmod k$?)

A quick example: Suppose we want to find the number of subsets of an n -element subset with number of elements divisible by 3. This is just the sum of the coefficients of

⁴For background on complex numbers see lecture 11, <http://onlinemathcircle.com/wp-content/uploads/2011/02/10-complex.pdf>

x^i (which are binomial coefficients $\binom{n}{i}$) for $3|i$, in $(1+x)^n$. Using Lemma 4.8 and letting $\omega = \frac{-1+\sqrt{3}i}{2}$ be the third root of unity, the answer is

$$\frac{2^n + (1 + \omega)^n + (1 + \omega^2)^n}{3} = \frac{2^n + \left(\frac{1+\sqrt{3}i}{2}\right)^n + \left(\frac{1-\sqrt{3}i}{2}\right)^n}{3}.$$

Now for something more hardcore.

Example 4.9 (IMO 1995/6): Let $p > 2$ be a prime number and let $A = \{1, 2, \dots, 2p\}$. Find the number of subsets of A having p elements and having sum of elements divisible by p .

Solution. Consider the generating function

$$(1+x)(1+x^2)\cdots(1+x^{2p}).$$

The coefficient of x^i in the expansion is the number of subsets of A whose sum of elements is i . Indeed, each subset of A is associated with a term in the expansion, by taking 1 from the i th term if $i \notin A$, and x^i from the i th term if $i \in A$.

Consider the generating function

$$\underbrace{(1+y)(1+y)\cdots(1+y)}_{2p}.$$

The coefficient of y^j in the expansion is the number of subsets of A with j elements. Indeed, each subset of A is associated with a term in the expansion, by taking 1 from the j th term if $j \notin A$, and y from the j th term if $j \in A$.

But we want information about both the number of elements in the subset and the sum of the elements in the subset, at the *same time*. What do we do? Put the two together! Consider the generating function

$$P(x, y) = (1+xy)(1+x^2y)\cdots(1+x^{2p}y).$$

The coefficient of $x^i y^j$ is the number of subsets of A with j elements with sum of elements equal to i . Think of the variables x and y as being counters for different things.

For example, in the case of $p = 3$, the term $xy \cdot 1 \cdot x^3y \cdot 1 \cdot x^5y \cdot 1 = x^9y^3$ corresponds to the subset $\{1, 3, 5\}$.

Thus we want the sum of the coefficients of $x^i y^p$ where $p|i$. Treating $P(x, y)$ as a polynomial in x and applying Lemma 4.8, we want the coefficient of y^p in

$$\frac{P(1, y) + P(\omega, y) + \cdots + P(\omega^{p-1}, y)}{p}, \quad \omega = e^{\frac{2\pi i}{p}}. \tag{11}$$

Note for $\zeta \neq 1$ a p th root of unity, $1, \zeta, \dots, \zeta^{p-1}$ runs through all p th roots of unity and hence $t^p - 1 = (t - 1)(t - \zeta) \cdots (t - \zeta^{p-1})$. Then

$$\begin{aligned} P(\zeta, y) &= [(1 + \zeta y) \cdots (1 + \zeta^p y)][(1 + \zeta^{p+1} y) \cdots (1 + \zeta^{2p} y)] \\ &= [(1 + \zeta y) \cdots (1 + \zeta^p y)]^2 \\ &= y^{2p} [(-y^{-1} - \zeta) \cdots (-y^{-1} - \zeta^p)]^2 \\ &= y^{2p} [(t^p - 1)|_{t=-y^{-1}}]^2 \\ &= (y^p + 1)^2. \end{aligned}$$

When $\zeta = 1$,

$$P(\zeta, y) = (y + 1)^{2p}.$$

Putting these two cases together,

$$\frac{P(1, y) + \cdots + P(\omega^{p-1}, y)}{p} = \frac{(y + 1)^{2p} + (p - 1)(y^p + 1)^2}{p}.$$

The coefficient of y^p is

$$\frac{\binom{2p}{p} + 2(p - 1)}{p}$$

which is the answer.

5 Problems

1. Find the generating function for the Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 0$. Then simplify the following (i.e. write it as an expression involving only a fixed number of F_i 's):

- (a) $\sum_{i=0}^n F_i$
- (b) $\sum_{i=0}^{n-1} F_{2i+1}$
- (c) $\sum_{i=0}^n F_{2i}$
- (d) $\sum_{i=0}^n iF_i$
- (e) $\sum_{i=0}^n F_i F_{n-i}$

2. Given that $\{a_n\}_{n=0}^{\infty}$ satisfies $\sum_{i=0}^n a_n p^{n-i} = b_n$ for each integer n , find a_n in terms of n and b_n . Same problem, for $\sum_{i=0}^n a_n (n-i)p^{n-i} = b_n$.

3. Let

$$(1 + x + x^2 + \cdots + x^8)^{100} = \sum_{n=0}^{800} a_n x^n.$$

Find $\sum_{n=0}^{800} n a_n$.

4. Prove that $\sum_{n=0}^m \binom{n+k}{k} = \binom{n+m+1}{k+1}$.
5. Given $n \in \mathbb{N}_0$, find the number of k -tuples of nonnegative integers (a_1, \dots, a_k) such that $a_1 + \cdots + a_k = n$.
6. Let $n > 1$ and consider a fair die D with n^2 faces, labeled $1, 2, \dots, n^2$. Prove that there exists two dice A and B , different from D , each with n^2 faces labeled by positive integers, such that the probability distribution of the sum of two rolls of D is the same as the probability distribution of the sum of one roll of A and one roll of B .
7. Let b_n be the number of ways to write n as a sum of nonnegative integer powers of 2.
 - (a) Find the generating function for b_n .

(b) Prove that

$$b_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_i.$$

Try to find both a combinatorial and a generating function proof, for comparison.

8. Let a_n denote the number of ways to tile a $3 \times n$ board with dominoes. Find a generating function for a_n , then give an explicit formula. (Note: you need to have two generating functions for this problem. One is for a_n ; one is for the b_n , the number of ways to tile a $3 \times n$ rectangle with one corner missing.)
9. Five people stand at the vertices of a pentagon, throwing frisbees to each other. At time 0, two people at adjacent vertices have the frisbees, and in each time interval, each frisbee is thrown to the left or right with probability $\frac{1}{2}$. This process continues until one person receives two frisbees simultaneously. Let p_n be the probability that the game ends after exactly n steps.
- (a) Find a generating function for p_n . (As in the previous problem, set up a system of equations involving generating functions.)
- (b) Find p_n .
- (c) Find the expected number of steps until the game ends ($\sum_{n=0}^{\infty} np_n$). Hint: differentiate. (This problem shows how generating functions can be useful in probability. See [2, §8.4] for a cool application of generating functions to coin-flipping games.)
10. (AwesomeMath Team Contest 2008, David Rush) If any element of the set

$$\{0, 1, \dots, m^3 - 1\}$$

is considered a digit in modified base m , prove that, for all positive integers n the number of modified base m representations of n is an integer multiple of $\frac{\lfloor \frac{n}{m^2} \rfloor + 1}{2}$.

11. Show that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.
12. (IMO 2008/5) Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled $1, 2, \dots, 2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider a sequence of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off. Let M be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine the ratio N/M .

13. Suppose that the natural numbers are written as a disjoint union of arithmetic progressions with differences $d_1 \geq d_2 \geq \dots \geq d_n$. Show $d_1 = d_2$.

14. (Euler's pentagonal number theorem) Prove that

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{k \in \mathbb{Z}} (-1)^k x^{k(3k-1)/2} = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots$$

Conclude that

$$p(n) = \sum_{k \in \mathbb{Z}} (-1)^{k-1} p\left(n - \frac{k(3k-1)}{2}\right).$$

(Hint: One way to proceed is to distribute the $(1-x)$ in each term, then distribute the $(1-x^2)$, and so on; proceed by induction after coming up with a suitable statement in terms of k . Combinatorial approaches are also possible.)

15. Show that

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = 1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)^2(1-x^2)^2 \dots (1-x^m)^2}.$$

(Hint: Given a partition, its corresponding *Ferrers diagram* is a left-justified array of dots with k rows, where there are λ_i dots in the i th row. The *Durfee square* of a Ferrers diagram is the largest square of dots, at the upper left hand corner of the diagram. How does the Durfee square split up the Ferrers diagram?)

16. (AwesomeMath Team Contest 2010, Brian Basham) Fatty McButterpants is bored and decides to toss stones into two buckets. Both buckets start with no stones, and every second he throws a stone into the first bucket, the second bucket, or no buckets, each with probability $\frac{1}{3}$. After n seconds, what is the probability that the number of stones in each bucket is divisible by 3?
17. (Mathematical Reflections, O107) Let p_1, p_2, p_3 be distinct primes and let n be a positive integer. Find the number of functions $f : \{1, 2, \dots, 2n\} \rightarrow \{p_1, p_2, p_3\}$ for which $f(1)f(2) \dots f(2n)$ is a perfect square.
18. How many subsets of $\{1, 2, \dots, 2000\}$ have sum of elements divisible by 5?
19. Let p and q be distinct primes greater than 2. How many subsets of size pq in $\{1, 2, \dots, 2pq\}$ have sum divisible by pq ?
20. (ISL 2002/A5) Let m and n be integers greater than 1 and let a_1, \dots, a_n be integers, none of which is divisible by m^{n-1} . Prove that we can find integers e_1, \dots, e_n , not all zero, such that $|e_i| < m$ for all i and $m^n | e_1 a_1 + \dots + e_n a_n$.
21. (Putnam 1992) For nonnegative integers n and k , define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^n \binom{n}{j} \binom{n}{k-2j}.$$

22. (Putnam 1997/B4) Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i, i} \leq 1.$$

23. (Putnam 1999/B3) Let $A = \{(x, y) : 0 \leq x, y < 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y) \rightarrow 1, (x,y) \in A} (1 - xy^2)(1 - x^2y)S(x, y).$$

24. (Putnam 2003/A6) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S, s_2 \in S, s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n ?

25. (Putnam 2005/B4) For positive integers m and n , let $f(m, n)$ denote the number of n -tuples (x_1, \dots, x_n) of integers such that $|x_1| + \dots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

26. Start with a finite number of chips placed on the integers on a number line; suppose c_i chips are placed on i . Let m and n be positive integers such that $n > m$. At each step, we may remove $m + n$ chips from an integer i and place 1 chip on $i + 1$ and mn chips on $i - 1$. Suppose for each integer i , there is a time after which the number of chips on i is constant. Let d_i be this constant. Prove that

$$\sum_{i \in \mathbb{Z}} c_i n^i = \sum_{i \in \mathbb{Z}} d_i n^i.$$

Is this true for n replaced by m ?

6 Appendix

Note the definition

$$\binom{m}{n} = \frac{m^n}{n!} = \frac{m(m-1) \cdots (m-n+1)}{n!}, \quad m \in \mathbb{R}, n \in \mathbb{N}_0.$$

Sequence	Generating function	Closed form
$1, 1, 1, \dots$	$\sum_{n=0}^{\infty} x^n$	$\frac{1}{1-x}$
$1, -1, 1, \dots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$\frac{1}{1+x}$
$1, r, r^2, \dots$	$\sum_{n=0}^{\infty} r^n x^n$	$\frac{1}{1-rx}$
$\underbrace{1, 0, \dots, 0}_k, 1, 0, \dots, 1, \dots$	$\sum_{n=0}^{\infty} x^{nk}$	$\frac{1}{1-x^k}$
$\binom{m}{0}, \binom{m}{1}, \dots, \binom{m}{m}, m \in \mathbb{N}$	$\sum_{n=0}^m \binom{m}{n} x^n$	$(1+x)^m$
$\binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \dots, r \in \mathbb{R}$	$\sum_{n=0}^{\infty} \binom{r}{n} x^n$	$(1+x)^r$
$\binom{r}{r}, \binom{r+1}{r}, \binom{r+2}{r}, \dots$	$\sum_{n=0}^{\infty} \binom{r+n}{r} x^n$	$\frac{1}{(1-x)^{r+1}}$
$1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \dots$	$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$	e^x
$1, 0, -\frac{1}{2!}, 0, \frac{1}{4!}, \dots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$\cos x$
$0, \frac{1}{1!}, 0, -\frac{1}{3!}, \dots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\sin x$
$1, 0, \frac{1}{2!}, 0, \frac{1}{4!}, \dots$	$\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$\cosh x = \frac{e^x + e^{-x}}{2}$
$0, \frac{1}{1!}, 0, \frac{1}{3!}, \dots$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$\sinh x = \frac{e^x - e^{-x}}{2}$
$0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$\ln(1+x)$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n, g(x) = \sum_{n=0}^{\infty} b_n x^n$.

Sequence	Generating function
$ca_n + b_n$	$cf + g$
a_{n-m}	$x^m f$
a_{n+m}	$\frac{f - a_{m-1}x^{m-1} - \dots - a_0}{x^m}$
$c^n a_n$	$f(cx)$
$a_{n/m}$ (0 if $m \nmid n$)	$f(x^m)$
$(n+1)a_{n+1}$	f'
na_n	xf'
$a_n b_0 + \dots + a_0 b_n$	fg
$a_0 + \dots + a_n$	$\frac{1}{1-x} f$
$\frac{a_{n-1}}{n}, n \geq 1$	$\int_0^x f(u) du$

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