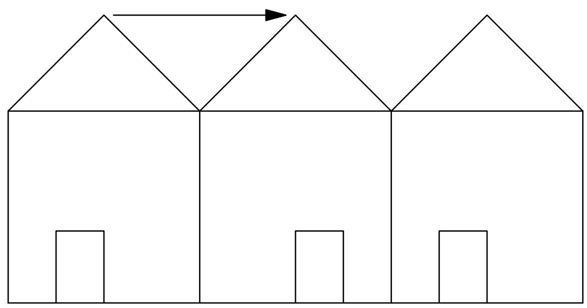
What is Congruence?

1 Hunting for Identical Triangles

Connie collects identical objects. She has five identical pencils in her pencil pouch, which she sharpens at the same time, to the same length. Every day to get to her bus stop, she walks past a row of townhouses, with identical triangular roofs. She knows they are identical because if she were to **shift** one of the triangles over one house, it would exactly be on top of the other triangle.



On a trip to the mountains with her friends, Connie comes across a wooden bridge, made up of triangles.

“That’s a lot of identical triangles.” Connie said, “Ten of them.”

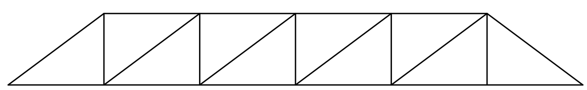
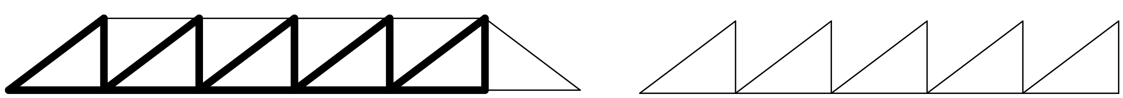
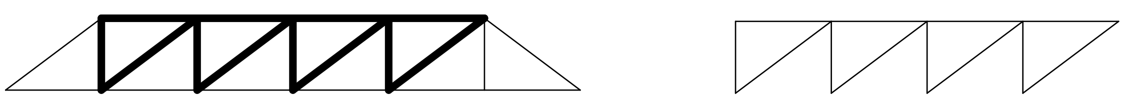


Figure 1: Why are these triangles the same?

Her friend John frowns. “They’re not all the same. Five of them are right-side up:”

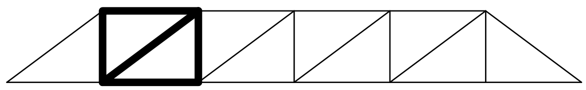


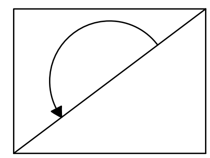
“Four of them are upside-down.”



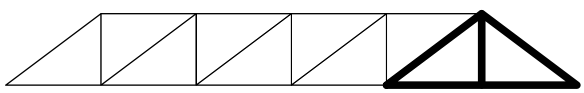
“And the one at the end is the wrong way round.”

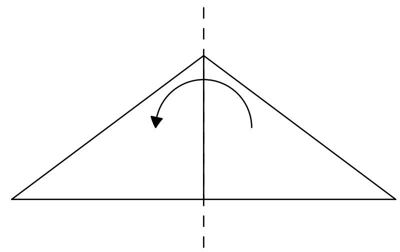
"But if you **turn** the upside-down one, you get the right-side up one," Connie says.





"And if you **flip** the last triangle, you get the right-side up one."





"So they are all identical triangles," she concludes.

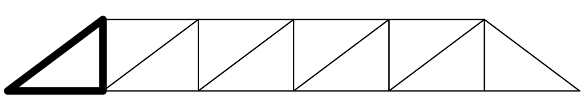
"That's interesting," John calls back. The rest of the class has gone on ahead, and he doesn't want to be left behind with the triangle girl. They haven't walked five minutes before they get to another bridge. "Where's the waterfall?" he grumbles. "So many rivers and no waterfall."

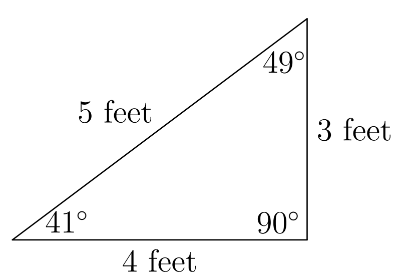
Again Connie stares at the bridge. "This bridge is the exact same shape as the previous one!"

"How do you know?" John taunts. "You can't move that bridge over here to check."

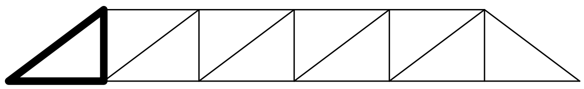
Connie thinks for a while. "That's true. But maybe I don't need to. I do have this, after all." She pulls a measuring tape and protractor out of her pocket. If I measure the sides and angles of the triangle and they're all the same, then I *know* if I were to move one of the triangles from the other bridge it would fit exactly on the triangle from this bridge."

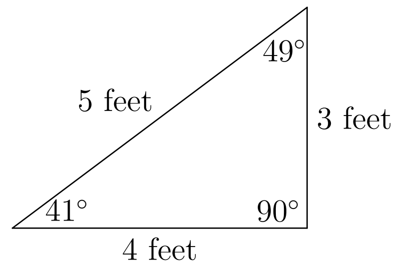
She measures one of the triangles on the bridge and finds that it lengths 4 feet, 3 feet, and 5 feet, and angles 49°, 41°, and 90°.





She then runs back to the first bridge, measures one of the triangles, and finds that is also has lengths 4 feet, 3 feet, and 5 feet, and angles 49°, 41°, and 90°.





“Aha, so the triangles are identical. This means that the two bridges are identical too!” It took a lot of time, though, to measure all the angles and sides of the triangles. “Is there a shorter way?” Connie thinks out loud. “Could I just have measured *some* of the angles and *some* of the sides?” If she measures all the sides and all the angles but one, that would be enough, she wonders.

Is this true? Could she do with less? Connie is quiet the rest of the hike as she ponders these questions.

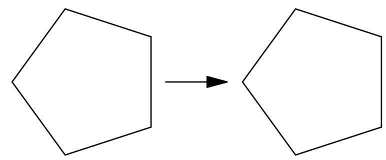
2 What it Means to be Identical

Like Connie, we are always on the hunt for objects that are identical. Mathematicians have a special name for "identical" objects.

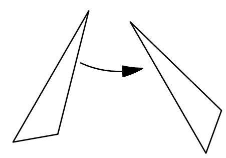
Two figures are **congruent** if we can move one of the polygons so that it lies on top of the other figure, and they match exactly. We are allowed to *shift* the figure, *rotate* the figure, or *reflect* the figure.

For example, the following pairs of figures are congruent:

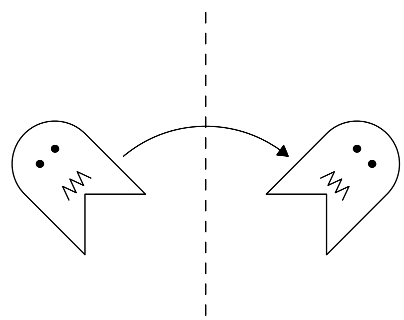
1. These pentagons are congruent because you can shift the one on the left to match the one on the right.



2. These triangles are congruent because you can rotate one of them so it matches the other.



3. These shapes are congruent because you can reflect the one on the left over the dashed line to match the one on the right.



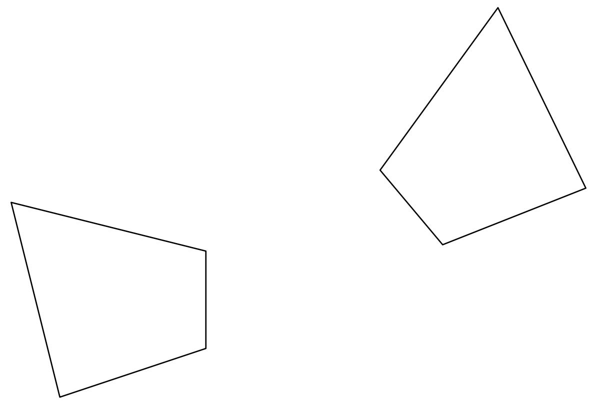
For now, we will mostly be concerned with congruent polygons. Polygons are especially simple figures because they consist only of *angles* and *straight edges*. It turns out that there is another way to know if two polygons are congruent or not, which we'll take as the "official" definition.

▶ **Definition:** Two polygons are **congruent** if we can label them in a way so that

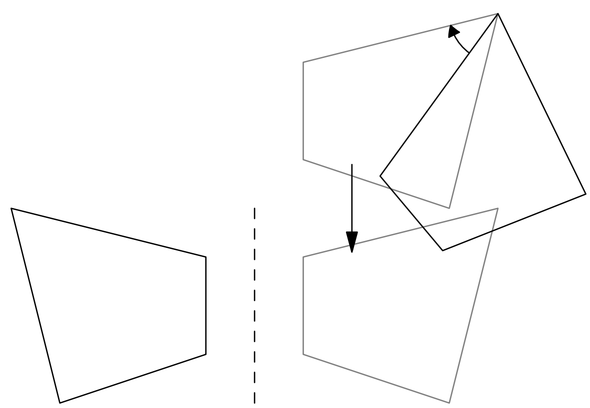
1. corresponding angles are equal, and
2. corresponding sides are equal.

As Connie understood, the second definition allows us to know figures are congruent even when it is not possible to mentally place one of them on the other.

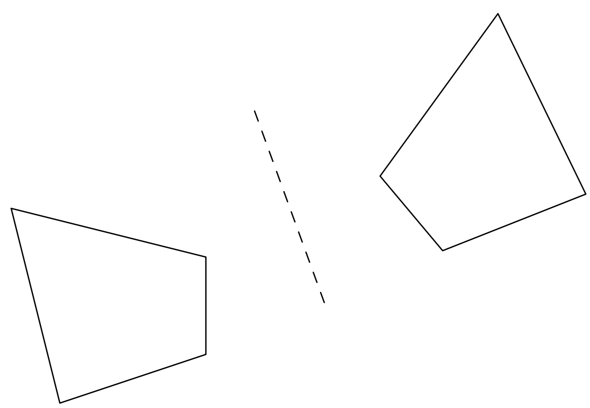
For example, the following two quadrilaterals are congruent.



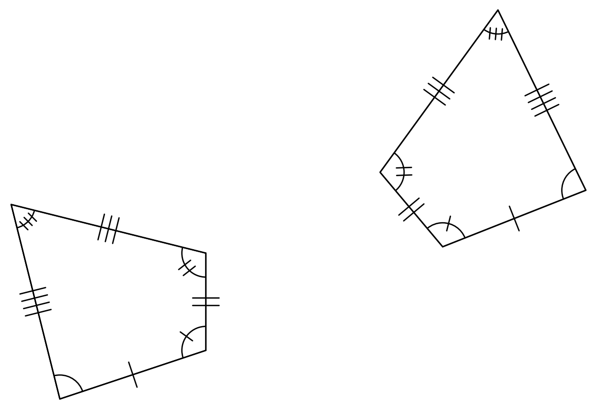
Note how one can be placed on another through a combination of shifts, rotations, and flips. One way to do this is through the sequence of moves below (it is not the only way).



A quicker way would be to refect the figure directly over the slanted line.



Rather than moving one of the figures onto the other, we can just verify all the sides and angles are equal by measuring.



Verify this is true by measuring with your ruler and protractor.

3 Notation

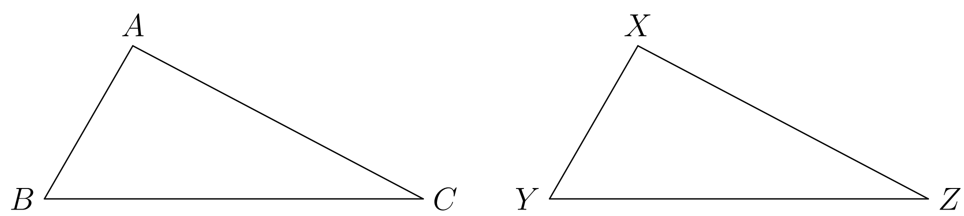
Now that we're making statements about congruence, we might as well simplify our language by introducing notation. We need to make sure our notation tells us "which points correspond to one another," in other words, if we were to move one polygon onto the other one, which points would coincide.

Congruence by corresponding points

To express the fact that two figures are congruent, we use the symbol $\cong$. For instance, the two triangles below are congruent, so we write

$$\triangle ABC \cong \triangle XYZ.$$

This statement reads "triangle ABC is congruent to triangle XYZ."



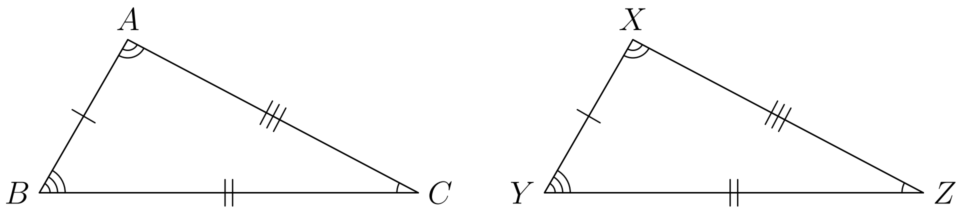
> Note that it is important that the we write the points in this order.

For instance, it is incorrect to say $\triangle ABC \cong \triangle YZX$, or $\triangle ABC \cong \triangle ZYX$, for instance. The statement $\triangle ABC \cong \triangle XYZ$ means *all of* the following.

1. $\angle A = \angle X$. $(\triangle \mathbf{A}BC \cong \triangle \mathbf{X}YZ.)$
2. $AB = XY$. $(\triangle \mathbf{AB}C \cong \triangle \mathbf{XY}Z.)$
3. $\angle B = \angle Y$. $(\triangle A\mathbf{B}C \cong \triangle X\mathbf{Y}Z.)$
4. $BC = YZ$. $(\triangle A\mathbf{BC} \cong \triangle X\mathbf{YZ}.)$
5. $\angle C = \angle Z$. $(\triangle AB\mathbf{C} \cong \triangle XY\mathbf{Z}.)$
6. $CA = ZX$. $(\triangle \mathbf{A}B\mathbf{C} \cong \triangle \mathbf{X}Y\mathbf{Z}.)$

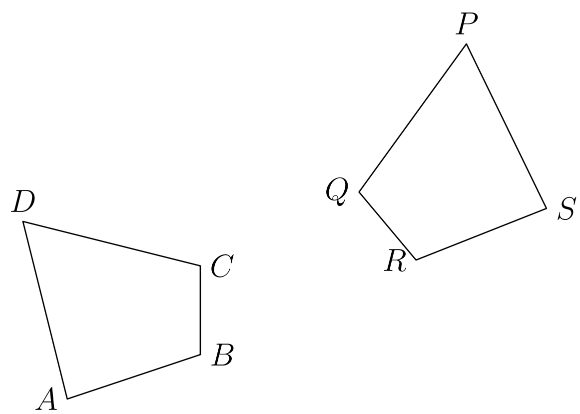
This principle is called **Corresponding Parts of Congruent Figures are Congruent**. "Congruent parts" include segments, angles, and curves. For triangles, we say **Corresponding Parts of Congruent Triangles are Congruent**, often abbreviated **CPCTC**.

> **CPCTC**: If two triangles are congruent, then all their corresponding parts are congruent.

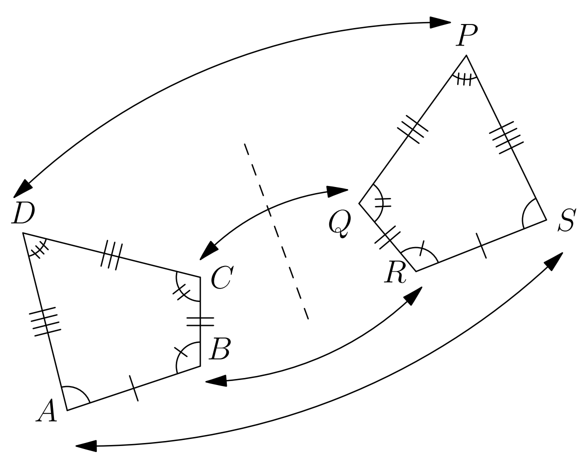


This notation is not limited to just triangles; we can apply it to figures with any number of sides.

Problem 1: How would you write the congruence statement for the following pair of figures?



Imagine moving $PQRS$ onto $ABCD$ so that the figures coincide. As we saw we can do this by reflection $PQRS$ through a slanted line. Under this reflection, P gets sent to D, Q gets sent to C, R gets sent to B, and S gets sent to A.



Hence one correct statement is

$$ABCD \cong SRQP.$$

Note that $BCDA \cong RQPS$, $CDAB \cong QPSR$, and $DABC \cong PSRQ$, are also fine, as are the backwards versions $DCBA \cong PQRS$, $CBAD \cong QRSP$, $BADC \cong RSPQ$, and $ADCB \cong SPQR$.

4 Certain Information is Enough for Congruence

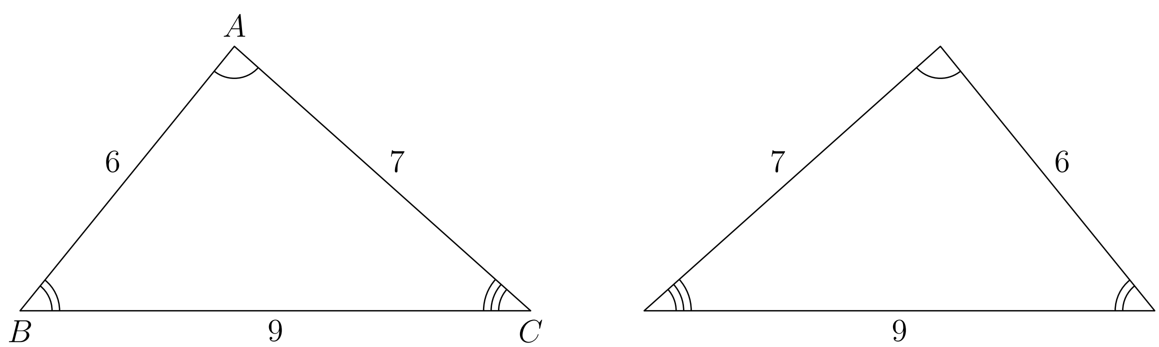
A triangle has 6 values associated with it: 3 side lengths and 3 angles. If we know that all 6 values—all side lengths and all angles—of 2 triangles are equal, then we know the triangles are congruent to one another.

But as Connie suggested, it would be nice not to have to know *all* the angles and sides are equal. Instead, we want the ability to conclude that two triangles are congruent given incomplete information—just some of the six values. Then, using CPCTC, we will get that *all* the segments and angles are equal, for free.

How many of these six values do we actually need, and does it matter which ones they are? To find out, keep reading.

5 Problem Solving with Congruence

1. Label the vertices of the triangle on the right with P, Q, and R so that $\triangle ABC \cong \triangle PQR$.



2. Suppose $\triangle ABC \cong \triangle YZX$, and we know

$$\begin{aligned} AB &= 8 & & \\ BC &= 15 & \qquad \angle B &= 60^\circ \\ CA &= 13 & \qquad \angle C &= 32^\circ. \end{aligned}$$

   Find all the side lengths and angles in triangle $\triangle YZX$.

3. To convince Connie that she can read minds, Connie's friend Karen asks Connie to draw a triangle and then attempts to copy the triangle. Connie writes down her measurements and gives it to Karen to compare.

   "I'm right!" Karen says. "Triangle CON is congruent to triangle KAR!"

   But before Connie can see if Karen's *actually* right, Eric drops his collection of malfunctioning ink pens on their paper, and they lose part of the data:

Connie	Karen
$CO = 24$	$AR =$ [redacted]
$ON = 27$	$RK = 20$
$NC =$ [redacted]	$KA =$ [redacted]
$\angle C =$ [redacted]	$\angle A = 45°$
$\angle O =$ [redacted]	$\angle R =$ [redacted]
$\angle N = 60°$	$\angle K =$ [redacted]

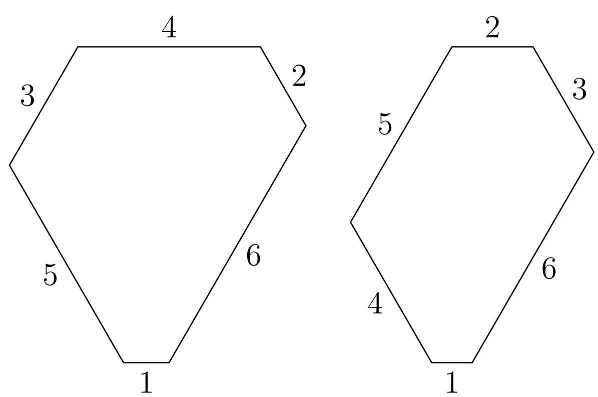
$$\triangle CON \cong \triangle KAR.$$

"Assuming Karen isn't lying," Eric says as he contemplates the mess he's made, "I can still tell you all the angles and lengths of both triangles."

What are all the angles and lengths?

4. Suppose that we know $\triangle ABC$ and $\triangle DEF$ are such that $AB = DE$, $BC = EF$, $CA = FD$, $\angle CAB = \angle FDE$, and $\angle ABC = \angle DEF$, so everything except 1 angle is known to be congruent. Does it follow that $\triangle ABC \cong \triangle DEF$?

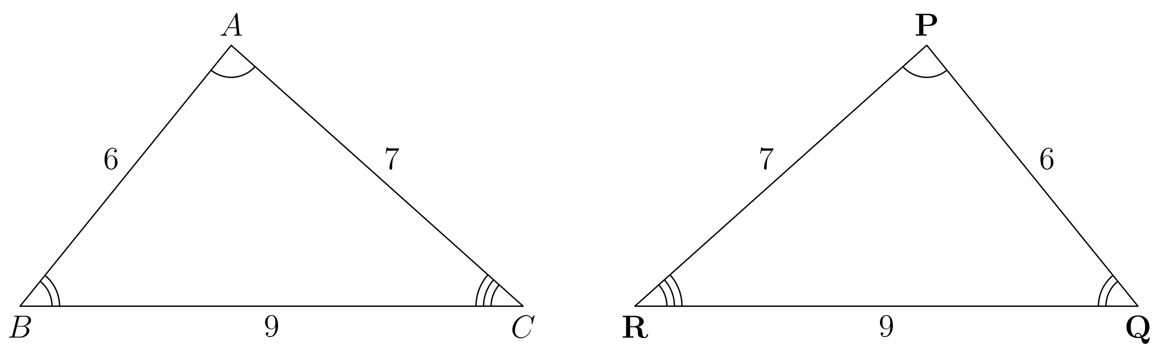
5. The following two shapes have all the same sides and all angles equal to 120°. Are they congruent?



6. See if you can answer Connie's question at the end: Could she just have measured *some* of the angles and *some* of the sides? How many measurements does she need to make? (We'll do this in the next few sections, but take a stab at it first!)

6 Lessons and Solutions

1. In order for $\triangle ABC \cong \triangle PQR$, we must have $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, $AB = 6 = PQ$, $BC = 9 = QR$, and $CA = 7 = RP$. So we label $\triangle PQR$ as follows.



2. We use the fact that corresponding parts of congruent triangles are congruent (CPCTC). The statement $\triangle ABC \cong \triangle YZX$ tells us that A corresponds to Y, B corresponds to Z, and C corresponds to X. Now we replace A by Y, B by Z, and C by X in

$$\begin{aligned} &AB = 8 && \\ &BC = 15 && \angle B = 60^\circ \\ &CA = 13 && \angle C = 32^\circ. \end{aligned}$$

to get

$$\begin{aligned} &YZ = 8 && \\ &ZX = 15 && \angle Z = 60^\circ \\ &XB = 13 && \angle X = 32^\circ. \end{aligned}$$

Finally, note that the sum of angles in a triangle is 180°, so

$$\angle Y = 180^\circ - 60^\circ - 32^\circ = 88^\circ.$$

3. We know that corresponding parts of $\triangle CON \cong \triangle KAR$ are congruent. So let's rewrite the data putting corresponding parts next to each other:

Connie	Karen
$CO = 24$	$KA = ?$
$ON = 27$	$AR = ?$
$NC = ?$	$RK = 13$
$\angle C = ?$	$\angle K = ?$
$\angle O = ?$	$\angle A = 45^\circ$
$\angle N = 60^\circ$	$\angle R = ?$

From this we can fill in the following lengths and angles.

Connie	Karen
$CO = 24$	$KA = 24$
$ON = 27$	$AR = 27$
$NC = 13$	$RK = 13$
$\angle C = ?$	$\angle K = ?$
$\angle O = 45°$	$\angle A = 45°$
$\angle N = 60°$	$\angle R = 60°$.

To find the remaining angle, we again use the fact that the angles in a triangle sum to 180°.

$$\angle C = \angle K = 180° - 45° - 60° = 75°.$$

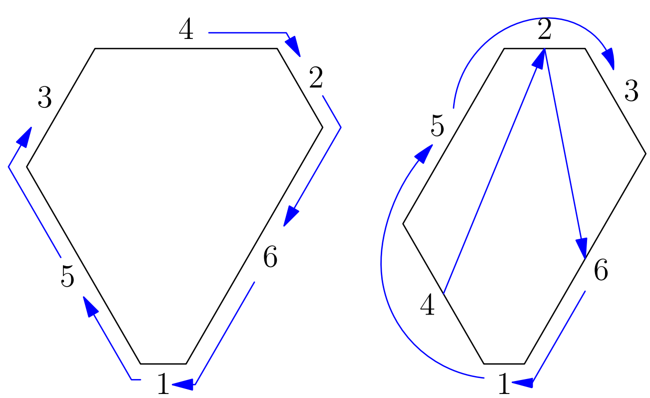
4. The given conditions tell us that the last angles are also equal, because the angles in a triangle sum to 180°:

$$\angle BCA = 180° - \angle CAB - \angle ABC = 180° - \angle FDE - \angle DEF = \angle EFD$$

All the sides and angles of $\triangle ABC$ and $\triangle DEF$ are equal, so by definition, $\triangle ABC \cong \triangle DEF$.

This problem shows how you can show two triangles are congruent just by knowing some of the parts are equal.

5. No, although both hexagons have 1 side of lengths 1, 2, 3, 4, 5, and 6, they are not in the correct order. There is no way to match up the hexagons so that the ordering of the sides in the hexagons are the same. Going around the left hexagon, we see sides of length 4, 2, 6, 1, 5, and 3, in that order. On the right hexagon, the sides of length 4, 2, 6, 1, 5, and 3 are not all adjacent.



The order of the sides and angles is important to congruence. Two figures are not congruent if the equal angles and sides don't appear in the same order.

6. Stay tuned!