What is Similarity?

1 Same Shape But Not the Same Size

Eric's dad is an airplane pilot. Whenever he comes home he brings home souvenirs of famous landmarks around the world, like the miniature cast-iron Eiffel Tower and miniature baked-clay pyramids that sit on the shelf besides his desk.

He dreams of flying a plane like his dad someday, but right now he isn't even old enough to drive yet. All he can do right now is make model airplanes. At least he is good at doing that: he might have inherited his dreams of flying from his dad, but he inherited the touch for making models from his mom. Fortunately, his mom doesn't mind him having a wood workshop in his room.

Eric has a little airport under his bed where he kept the planes he has already built: a Boeing 747 sits by an unmanned bombers and a fighter jet. He is going to try something new today, though. He takes out the sketchpad he had brought to the National Air and Space Museum, where he had drawn a detailed diagram of Wright Flyer, the first airplane ever built by humans.

He tapped the wood against the table, unsure where to begin. "Drat," he said, "I drew it at an angle so I don't know how long to make the wings versus the body. Or how long versus how tall it is." He draws several sketches but can't decide which one looks better than the others. After all, planes back then were different.

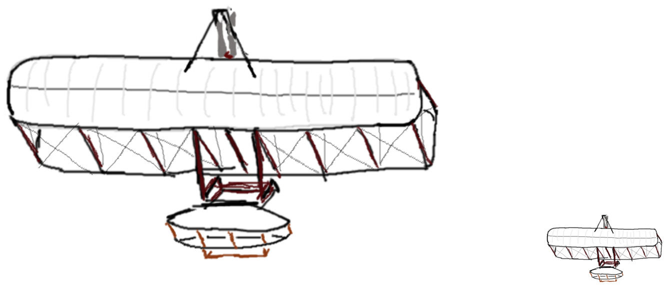
Just then his dog Cumulus bounds into the room and jumps on a chair, then onto the table, by the computer. (Like Eric, he likes high places.) "Go float somewhere else," he says, putting Cumulus down on the floor. "Oh, that's right," he suddenly thinks, "I'll just have to look up the dimensions." He looks them up on Wikipedia and rounds them to the nearest feet, to make things easy.

Dimension	Actual size
Length	21 feet
Wingspan	40 feet
Height	9 feet

Too big to make, he thought, *although I'd love to make a full-scale plane I could actually fly.* He'd have to shrink everything by the same amount. How about 20 inches for the wingspan?

Dimension	Actual size	Model size
Length	21 feet	
Wingspan	40 feet	20 inches
Height	9 feet	

Problem 1: Eric wants his model Wright Flyer to be the same shape as the original. If he makes the the wingspan of his model Wright Flyer 20 inches, how long and how high should he make it?



How much did Eric shrink the Wright Flyer going from 40 feet to 20 inches? He shrunk 1 foot to 1 inch, and then shrunk it by a factor of $\frac{1}{2}$. This means that he has to shrink the length and height by the same amount. We have

$$21 \cdot \frac{1}{2} = 10.5, \qquad 9 \cdot \frac{1}{2} = 4.5.$$

We get the following dimensions.

Dimension	Actual size	Model size
Length	21 feet	10.5 inches
Wingspan	40 feet	20 inches
Height	9 feet	4.5 inches

Note how every dimension is $\frac{1}{24}$ of the actual size:

$$\begin{array}{cc} \text{Length:} & \frac{10.5}{21}\frac{\text{inch}}{\text{feet}} = \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24} \\ \text{Wingspan:} & \frac{20}{40}\frac{\text{inch}}{\text{feet}} = \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24} \\ \text{Height:} & \frac{4.5}{9}\frac{\text{inch}}{\text{feet}} = \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24}. \end{array}$$

2 Find the Height of a Tree

Later that day, Connie calls. “Eric, do you want to join us? We’re measuring trees,” she said.

“What?” Measuring trees doesn’t sound as interesting as making planes.

“Leo wants his dad to build a tree house and his dad says he’ll do it if Leo can find the height of the tallest tree in the park. Leo says we can share his tree house if we help. And give us ice cream.” She lowers her voice. “I think we can do it. Seeing as we’re good at geometry and all.”

Having finished his plane, Eric agrees to give tree-measuring a shot. Besides, Cumulus is feeling restless; it’s about time to take him on a walk.

In the park, Eric and Connie watch as Leo and Alan try to climb the tree. They have duct-taped one end of the tape measure to the ground. The other end is attached to Leo’s belt, and he is kind of stuck because the tape measure has run out.

“You two aren’t helping,” Alan calls down. “Are you scared of climbing trees?”

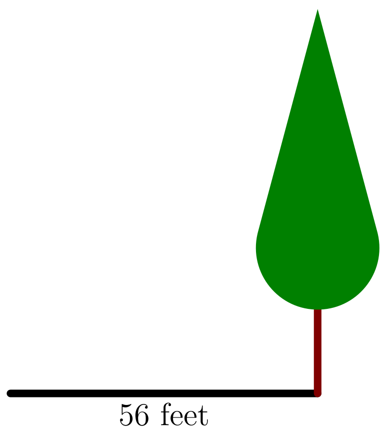
Connie looks doubtfully at the twosome. They aren’t even a third of the way up the tree. “I think we should be able to solve this without climbing, don’t you think, Eric?”

Eric thinks for a while. “We can’t measure the tree directly,” she says, “but we can measure the tree’s *shadow*.”

Connie measures the shadow with her own tape measure and finds that it is 56 feet long.

“How does that help us, though? The shadow isn’t the same length as the tree.”

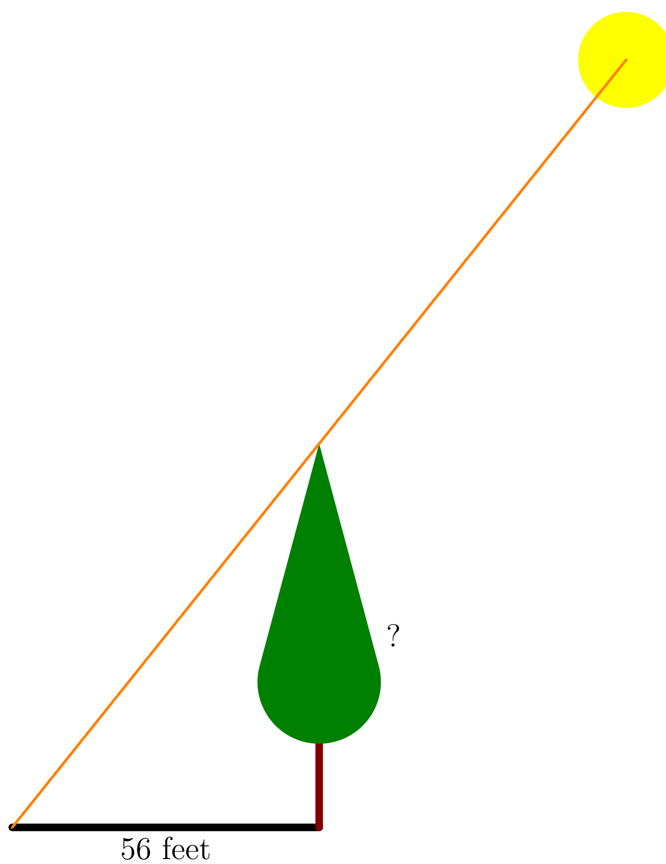
Eric draws a picture on his sketchpad.



“That looks like a triangle!” Connie says.

“I don’t see any triangle.” Connie was always seeing triangles in odd places.

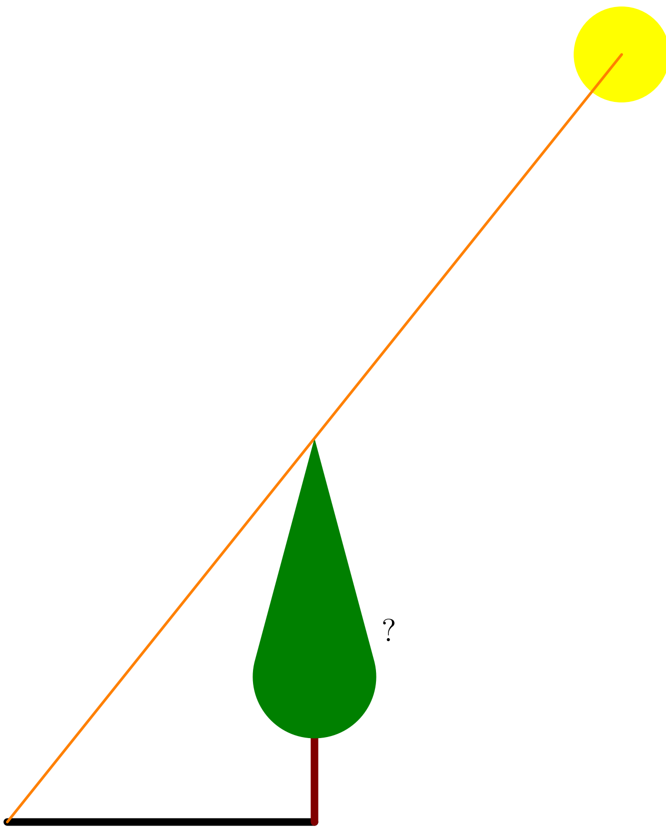
Connie draws in a sunray.

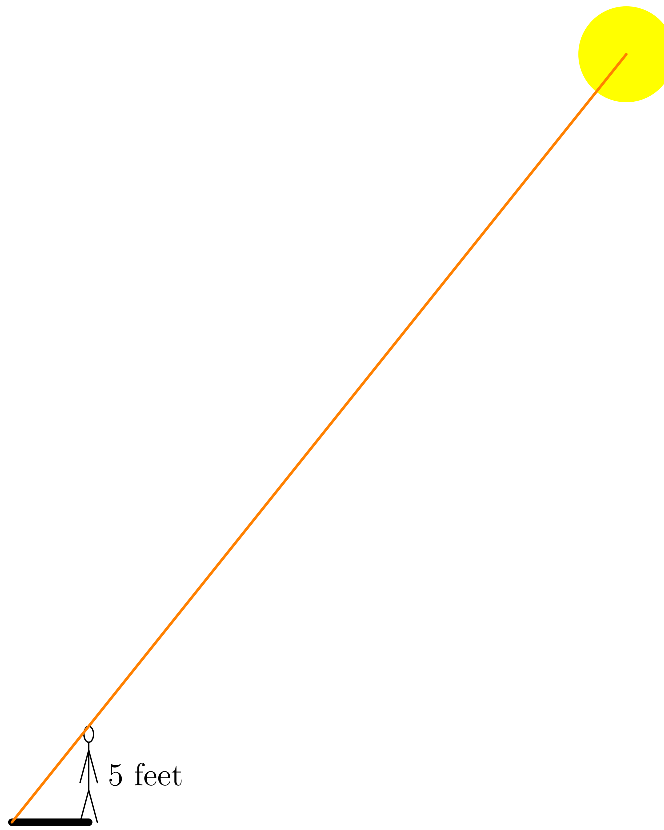


“Now if we can only find a congruent triangle...” Connie looks around her and frowns.

“Even if we had one, it would be too big to measure,” Eric says. *Too big to measure.* Just like the Wright Flyer, too big to make. “We could make a smaller triangle... that’s it! Measure the length of my shadow.”

“4 feet,” she said. She peeks over Eric’s shoulder at the sketchpad. “Congruent triangles! Except they’re not the same size.”



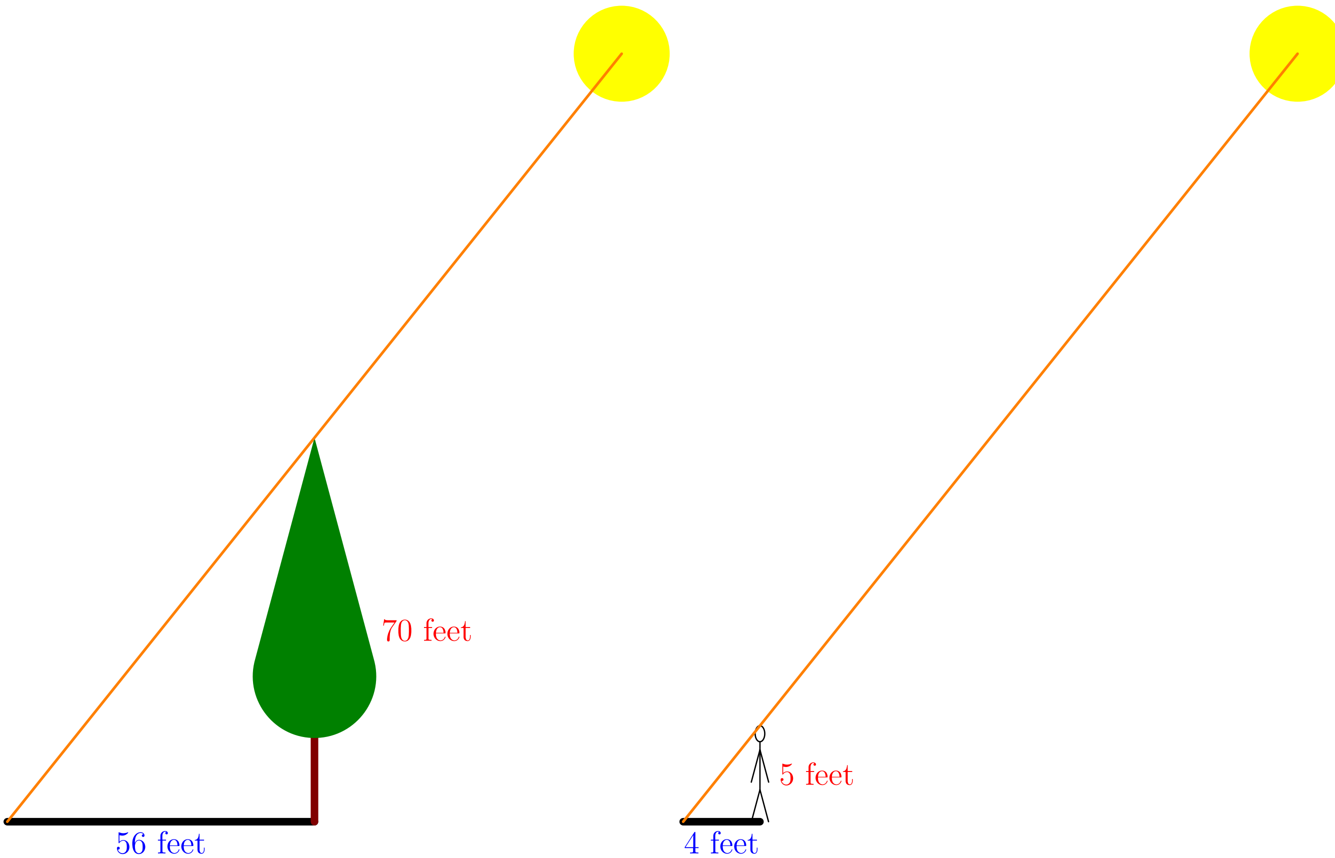


Eric makes a table, just like he did for the airplane:

	Shadow	Height
Tree	56 feet	?
Eric	4 feet	5 feet

Problem 2: How tall is the tree?

“Since the tree’s shadow is 14 times as long as my shadow,” Eric says, “the tree must be 14 times as tall. So the tree is $5 \cdot 14 = 70$ feet tall. Connie?”



$$\frac{56 \text{ feet}}{4 \text{ feet}} = \frac{70 \text{ feet}}{5 \text{ feet}} = 14.$$

Connie is still staring at the triangles. “I think,” she said slowly, “We need a new name for these kinds of triangles. And we need a new Theory for them. Like for congruent triangles.”

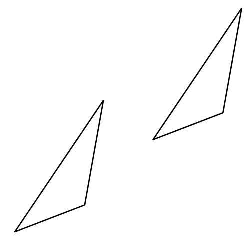
3 What It Means to be Similar

If we only looked for objects that are identical, or *congruent*, to one another, we miss out on a lot. Sometimes, we just care about objects that are the same shape, and not the same size.

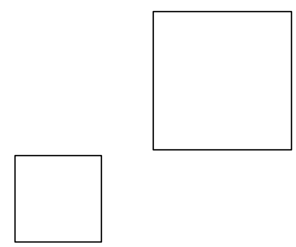
> Two figures are **similar** if we can be rotate, shift, flip, and/or *scale* one of them so that it lies on top of the figure, and they match exactly.
>
> In other words, we can scale one of the figures so that it becomes *congruent* to the other one.

For example, the following pairs of figures are similar.

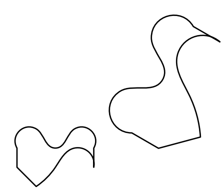
1. These two triangles are similar to each other because one is just a translation of the other. In fact, they are congruent to each other, since they are the same size. We can think of congruence as just a special case of similarity.



2. These two quadrilateral are similar because the larger is just a rescaling of the smaller.



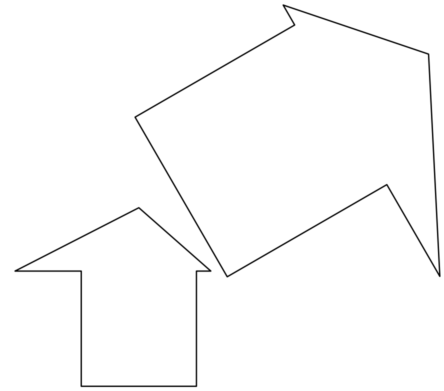
3. These two figures are similar because the larger figure is the smaller figure scaled and rotated.



Polygons are especially simple figures because they consist only of *angles* and *straight edges*. Just like with congruence, there is another way to tell if two polygons are similar.

For congruence, we wanted corresponding sides to be equal. For similarity, the sides of one polygon can be larger or smaller than the other – but they have to be larger or smaller *by the same factor*.

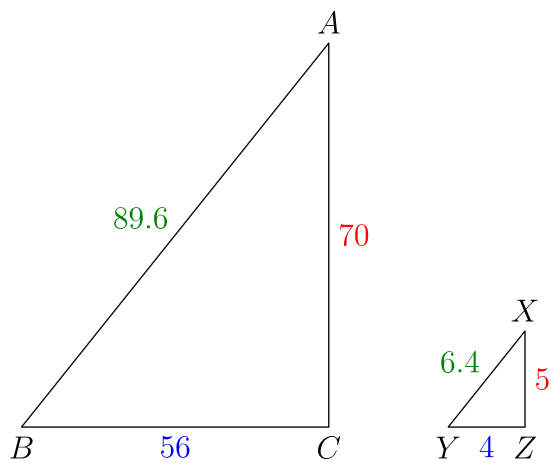
4. These two polygons are similar because the larger polygon is the smalller polygon scaled, rotated, and then reflected.



> ▶ **Definition:** Two polygons are similar if we can label them in a way so that
>
> 1. corresponding angles are equal, and
> 2. corresponding sides are in a constant ratio.

3.1 Constant Ratio

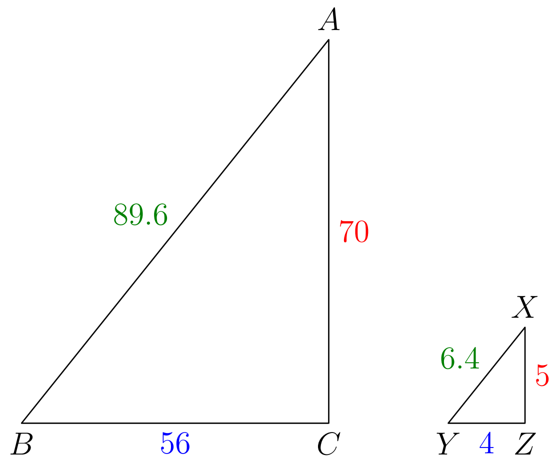
Note that we have two ways to think about what "constant ratio" means. The first is just that we multiply every length in the one figure by the same factor to get the lengths in the other figure.



$$\frac{56}{4} = \frac{70}{5} = \frac{89.6}{6.4} = 14.$$

Here we are multiplying by 14.

The second is that *within* each figure, we multiply by the same factor to get from one side length to another.



$$\frac{70}{56} = \frac{5}{4} = 1.25$$
$$\frac{89.6}{56} = \frac{6.4}{4} = 1.6$$
$$\frac{89.6}{70} = \frac{6.4}{5} = 1.28.$$

We multiply by 1.25 to get from the blue side to the red side, and similarly for the other sides.

Note the reason that we have two ways to think about "constant ratio" is just because we have 2 ways to write any proportion.

> There are two ways to write any proportion:
>
> $$\frac{a}{b} = \frac{x}{y} \qquad \Longleftrightarrow \qquad \frac{a}{x} = \frac{b}{y}.$$
>
> This means that there are two ways we write the proportions for triangle similarity.

To encapsulate the idea that we can view ratios in 2 ways, we will often write the fact that several quantities are in constant ratio as

$$56 : 70 : 89.6$$
$$= 4 : 5 : 6.4.$$

As we saw, we can interpret this as

$$\frac{56}{4} = \frac{70}{5} = \frac{89.6}{6.4}$$

or as three separate proportions (we gray out the side length not considered in each proportion)

$$\boxed{56 : 70} : 89.6 \qquad \boxed{56} : 70 : \boxed{89.6} \qquad 56 : \boxed{70 : 89.6}$$
$$= \boxed{4 : 5} : 6.4 \qquad = \boxed{4} : 5 : \boxed{6.4} \qquad = 4 : \boxed{5 : 6.4}$$
$$\frac{70}{56} = \frac{5}{4} \qquad \frac{89.6}{56} = \frac{6.4}{4} \qquad \frac{89.6}{70} = \frac{6.4}{5}.$$

For instance, make sure you can see all the proportions in the following:

$$\begin{aligned} &1:2:3 \\ =\ &2:4:6 \\ =\ &3:6:9 \\ =\ &4:8:12. \end{aligned}$$

We'll concentrate mainly on similar triangles, just like we concentrated on congruent triangles, because triangles are the building blocks of all polygons.

4 Notation

Now that we are talking about similar polygons, we will need some new notation. Just like with congruent polygons, our notation tells us which points correspond to one another. The notation is very similar to the notation for congruent polygons. We write

$$\triangle ABC \sim \triangle XYZ$$

to denote that triangle ABC is similar to triangle XYZ.

The statement $\triangle ABC \sim \triangle XYZ$ means *all* of the following.

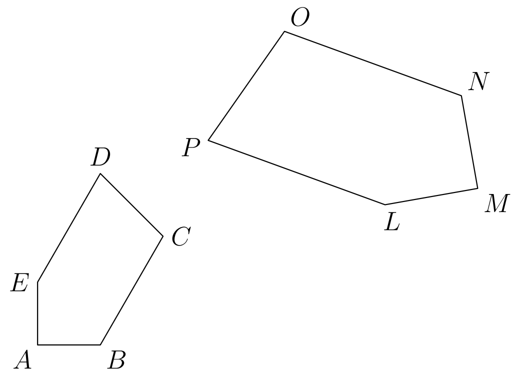
$$\begin{aligned} \angle A = \angle X &\quad (\triangle \mathbf{A}BC \sim \triangle \mathbf{X}YZ) \\ \angle B = \angle Y &\quad (\triangle A\mathbf{B}C \sim \triangle X\mathbf{Y}Z) \\ \angle C = \angle Z &\quad (\triangle AB\mathbf{C} \sim \triangle XY\mathbf{Z}) \\ \frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = k & \end{aligned}$$

The last chain of equalities tells us that the corresponding sides are in a constant ratio, and that ratio k is called the **similarity ratio** between $\triangle ABC$ and $\triangle XYZ$. We can rewrite that ratio as

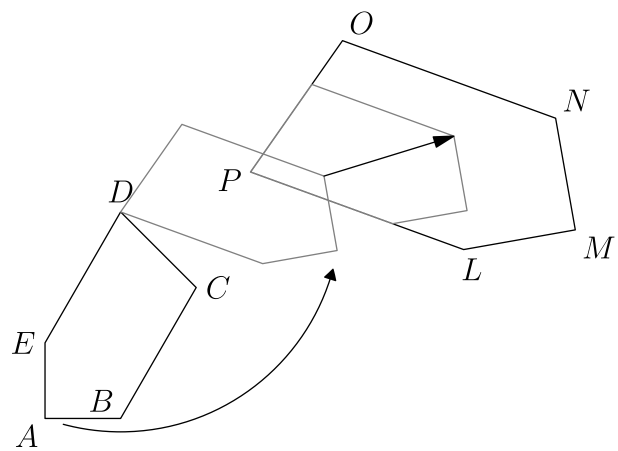
$$XY : YZ : ZX = AB : BC : CA$$

which tells us that the ratio of respective sides within a triangle is the same in similar triangles.

Problem 3: How would you write the similarity statement for the following two polygons?



We can get the figure on the right if we rotate the figure on the left counterclockwise by about 100 degrees and then shift it to the right and upwards, and then scale by 1.5.



We see that vertex A gets sent to vertex M, B gets sent to N, C gets sent to O, D gets sent to P, and E gets sent to L. We write the similarity statement

$$ABCDE \sim MNOPL.$$

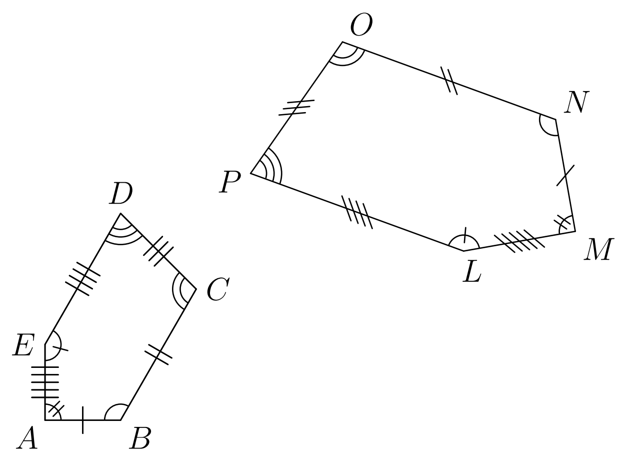
You could have also written something like $EABCD \sim LMNOP$ or $EDCBA \sim LPONM$, as long as A corresponds to M, B corresponds to N, and so forth, and you go in order around the polygon.

Another way to see similarity is to measure all the angles and sides. We find that corresponding angles are equal,

$$\angle A = \angle M = 90^\circ \qquad \angle D = \angle P = 75^\circ$$
$$\angle B = \angle N = 120^\circ \qquad \angle E = \angle L = 150^\circ$$
$$\angle C = \angle O = 105^\circ$$

and corresponding sides are proportional (measurements are in centimeters)

$$AB : BC : CD : DE : EA = 1 : 2 : 1.4 : 2 : 1$$
$$= MN : NO : OP : PL : LM = 1.5 : 3 : 2.1 : 3 : 1.5.$$



Again we see $ABCDE \sim MNOPL$.

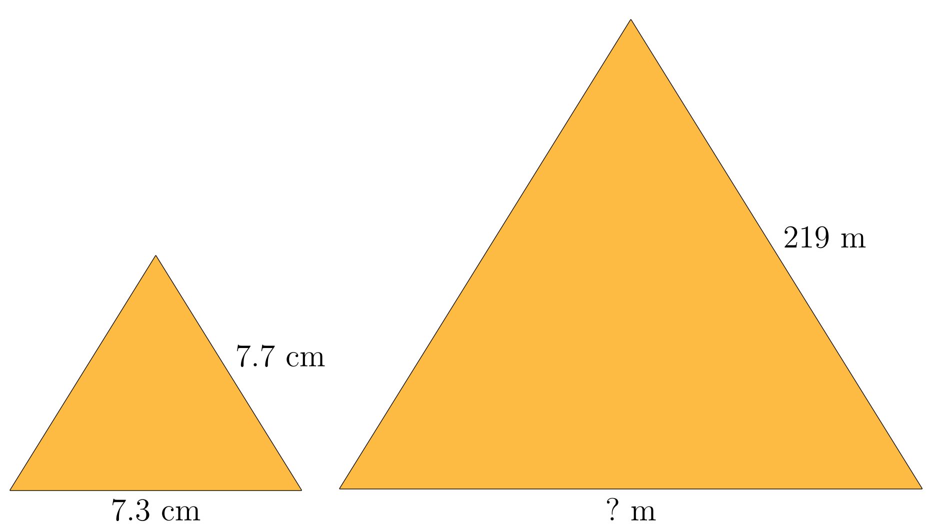
5 Certain Information is Enough for Similarity

We know that if two triangles are similar, then we have six pieces of information. We know that three ratios of corresponding sides are all the same and that the three corresponding angles are all the same. But again, just like with congruent triangles, we want to know if we can tell two triangles are similar with less information. In the next four sections, we will discuss exactly what information we need to conclude two triangles are similar.

6 Problem Solving with Similiarity

Eric measures cast-iron Eiffel Tower and clay pyramid, to see how much smaller they are than the real thing.

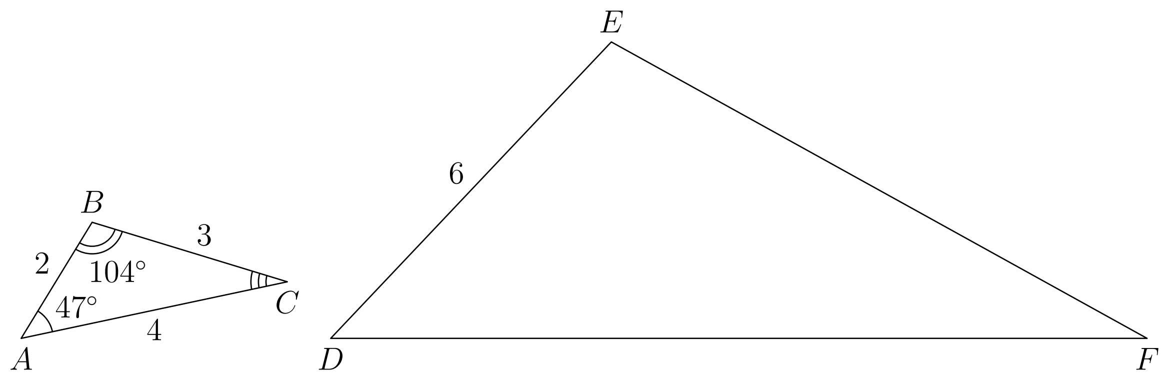
1. Eric measures his Eiffel Tower to be 5 centimeters wide and 13 centimeters tall. Given that the real Eiffel Tower is 2500 times as large, how wide and how tall is the Eiffel Tower?
2. Eric finds that each face of his clay pyramid is an isosceles triangle, whose legs are 7.3 cm long and whose base is 7.7 cm wide. One of the legs of the Great Pyramid of Giza is 219 m long.



(a) How much larger is the Great Pyramid of Giza? (In other words, what is the **similarity ratio**?)

(b) How wide is the base of the Great Pyramid of Giza?

3. In the picture, $\triangle ABC$ has $AB = 2$, $BC = 3$, $CA = 4$, $\angle A = 47°$, and $\angle B = 104°$. Given that $\triangle DEF$ has $DE = 6$, find all the angles and side lengths of $\triangle DEF$.



4. Suppose $\triangle DOG \sim \triangle FLY$, and we know

$$DO = 8 \qquad \angle D = 44°$$
$$GD = 14 \qquad \angle Y = 34°$$
$$LY = 15$$
$$FY = 21$$

Find all the angles and sides of $\triangle DOG$ and $\triangle FLY$.

5. Suppose $ABCDE$ is a pentagon with $AB = 3$ and perimeter 12. Given that $ABCDE \sim ZYXWV$ and $ZY = 24$, what is the perimeter of $ZYXWV$?

6. Given that $\triangle ABC \sim \triangle DEF$, that $AB = 5$, $BC = 6$, and $CA = 8$, and that *one* of the sides of $\triangle DEF$ is equal to 120, what are the maximum and minimum possible values for the perimeter of $\triangle DEF$?

7 Lesson and Solutions

1. To find how large the real Eiffel Tower is, we multiply the lengths in the model by the similarity ratio. The actual dimensions are

$$\begin{aligned} \text{Width:} \quad & 5 \text{ cm} \cdot 2500 = 12500 \text{ cm} = \boxed{125 \text{ m}} \\ \text{Height:} \quad & 13 \text{ cm} \cdot 2500 = 32500 \text{ cm} = \boxed{325 \text{ m}} \end{aligned}$$

   (We're guilty of some slight rounding here; the actual width is 124.9m and the actual height is 324m.)

2. This time we don't know what the similarity ratio is, so we have to calculate it first.

   (a) To find the similarity ratio, we divide a corresponding length of the big pyramid by a corresponding length of the small pyramid.

$$\frac{219 \text{ m}}{7.3 \text{ cm}} = \frac{219}{7.3} \frac{\text{m}}{\text{cm}} = 30 \cdot 100 = \boxed{3000}$$

   The actual pyramid is 3000 times as large as the model.

   (b) Since the actual pyramid is 3000 times the size of the model, the base of the actual pyramid is

$$3000 \cdot 7.7 \text{ cm} = 23100 \text{ cm} = \boxed{231 \text{ m}}.$$

To find the lengths in a similar figure,

- first find the similarity ratio, and
- then multiply each length by this ratio.

3. This problem is phrased in math language but the idea is the same. Side DE corresponds to side AB, and the ratio between these lengths is

$$\frac{DE}{AB} = \frac{6}{2} = 3.$$

   The similarity ratio is 3.

   Now we multiply each length by 3 to find

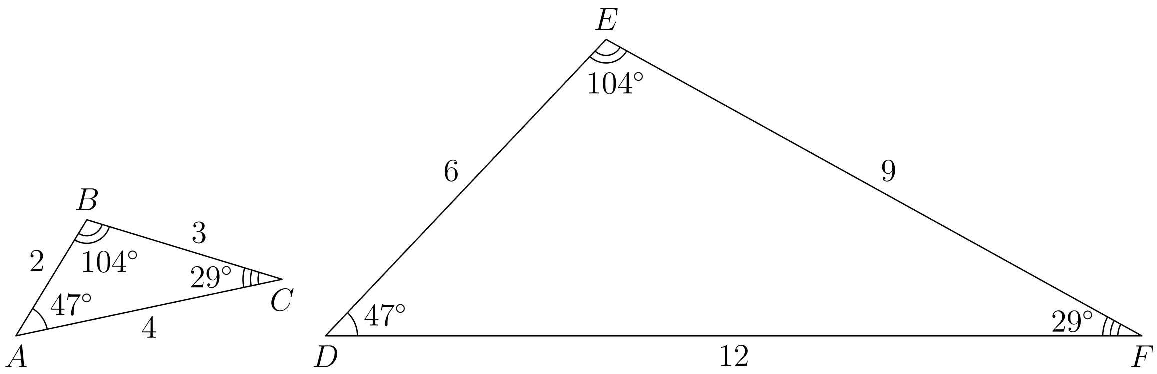
$$\begin{aligned} EF &= 3 \cdot 3 = \boxed{9}, \\ DF &= 3 \cdot 4 = \boxed{12}. \end{aligned}$$

   Just like in congruence, the angles in $\triangle DEF$ are the same as the corresponding angles in $\triangle ABC$:

$$\begin{aligned} \angle D &= \angle A = \boxed{47^\circ} \\ \angle E &= \angle B = \boxed{104^\circ}. \end{aligned}$$

   Finally, to find $\angle F$ we use the fact that the angles in a triangle add up to 180°.

$$\angle F = \angle C = 180^\circ - 47^\circ - 104^\circ = \boxed{29^\circ}$$



4. We first organize our data, by putting corresponding sides and angles of $\triangle DOG$ and $\triangle FLY$ together.

$$\begin{array}{ll} DO = 8 & FL = ? \\ OG = ? & LY = 15 \\ GD = 14 & YF = 21 \\ \angle D = 44^\circ & \angle F = ? \\ \angle O = ? & \angle L = ? \\ \angle G = & \angle Y = 34^\circ. \end{array}$$

Dividing the length of corresponding sides, we see that the similarity ratio is

$$\frac{YF}{GD} = \frac{21}{14} = \frac{3}{2}.$$

This means that

$$FL = \frac{3}{2} \cdot DO = \frac{3}{2} \cdot 8 = 12.$$

How do we calculate OG, a side length in the *first* triangle? Since we multiply by $\frac{3}{2}$ to get from the lengths in the first triangle to the lengths in the second triangle, we divide by $\frac{3}{2}$ (i.e. multiply by $\frac{2}{3}$) to get from the lengths in the second triangle to the lengths in the first triangle.

$$OG = \frac{2}{3} \cdot 15 = 10.$$

Corresponding angles of similar triangles are equal, and the remaining angle is equal to

$$\angle O = \angle L = 180^\circ - 44^\circ - 34^\circ = 102^\circ.$$

We summarize the results.

$$\begin{array}{ll} DO = 8 & FL = \mathbf{12} \\ OG = \mathbf{10} & LY = 15 \\ GD = 14 & YF = 21 \\ \angle D = 44^\circ & \angle F = \mathbf{44}^\circ \\ \angle O = \mathbf{102}^\circ & \angle L = \mathbf{102}^\circ \\ \angle G = \mathbf{34}^\circ & \angle Y = 34^\circ. \end{array}$$

5. As before, we first find the similarity ratio of the figures:

$$\frac{ZY}{AB} = \frac{24}{3} = 8.$$

We don't know any of the lengths, but let's think about what it means for the similarity ratio to be 8. This means that *every* side of $ZYXWV$ is 8 times the length of the corresponding side of $ABCDE$. Since the perimeter is just the sum of all the side lengths, this means that the perimeter of $ZYXWV$ is 8 times the perimeter of $ABCDE$:

$$8 \cdot 12 = \boxed{96}.$$

If we scale a figure by a constant k, then the perimeter scales by the same constant k.

6. First, we find the perimeter of the smaller figure: $5 + 6 + 8 = 19$.

We're not given which side has length 120, so it could correspond to the sides of length 5, 6, or 8.

- If we want to *maximize* the perimeter of $\triangle DEF$, we need to maximize the similarity ratio. The similarity ratio is largest when 120 corresponds to the shortest side 5; it is $\frac{120}{5} = 24$. Then the perimeter of $\triangle DEF$ is 24 times the perimeter of $\triangle ABC$:

$$24 \cdot 19 = \boxed{456}.$$

- If we want to *minimize* the perimeter of $\triangle DEF$, we need to minimize the similarity ratio. The similarity ratio is largest when 120 corresponds to the longest side 8; it is $\frac{120}{8} = 15$. Then the perimeter of $\triangle DEF$ is 15 times the perimeter of $\triangle ABC$:

$$15 \cdot 19 = \boxed{285}.$$