Solutions to Lecture 8 — Polynomials

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1 Values and Zeros

1. This time it's easier to guess the solution. The polynomial

$$Q(x) = \sum_{i=0}^{n} \binom{x}{i} (r-1)^i$$

has degree n and by the Binomial Theorem, satisfies the given conditions. Since $P(x) = Q(x)$ for $n+1$ values of x, actually P, Q are the same polynomial, and

$$P(n+1) = Q(n+1) = \left(\sum_{i=0}^{n+1} \binom{n+1}{i} (r-1)^i \right) - (r-1)^{n+1} = r^{n+1} - (r-1)^{n+1}.$$

2. Letting ray OQ_1 be the positive real axis, Q_i represent the nth roots of unity ω^i in the complex plane. Hence PQ_i equals $|2 - \omega^i|$. The roots of $x^n - 1 = 0$ are just the nth roots of unity, so $x^n - 1 = \prod_{i=0}^{n-1} (x - \omega^i)$. Plugging in $x = 2$ gives $\prod_{k=1}^{n} |PQ_i| = 2^n - 1$.

3. The given condition says

$$f(x)^2 = x(x-1)\cdots(x-n)Q(x) + x^2 + 1 \tag{1}$$

for some polynomial $f(x)$ of degree at most n. Plugging $x = 0, 1, \ldots, n$ into (1) gives

$$f(x) = \pm\sqrt{x^2+1}, \text{ when } x = 0, 1, \ldots, n. \tag{2}$$

The following is key: Given $n+1$ points $(x_0, y_0), \ldots, (x_n, y_n)$ with distinct x-coordinates, there exists exactly one polynomial f of degree at most n so that $f(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

Applying this to (2) we get 2^{n+1} possibilities for $f(x)$ since we have 2 choices of sign for each of $x = 0, 1, \ldots, n$. If $f(x)$ is a solution to (2) then so is $-f(x)$; we get 2^n possibilities for $f(x)^2$. Solve (1) to get 2^n possibilities for $Q(x)$:

$$Q(x) = \frac{f(x)^2 - x^2 - 1}{x(x-1)\cdots(x-n)}$$

Each such polynomial is a valid solution because $f(x)^2 - x^2 - 1$ is zero at $x = 0, 1, \ldots, n$ and hence is divisible by $x(x-1)\cdots(x-n)$.

4. Clearing denominators,

$$\sum_{i=1}^{5}\left[a_i x \prod_{i\neq j, 1\leq j\leq 5}(x+j)\right]-(x+1)(x+2)(x+3)(x+4)(x+5)=0$$

for $x = 1, 4, 9, 16, 25$. Let $f(x)$ denote the LHS. Since $f(x) = 0$ has the roots $x = 1, 4, 9, 16, 25$, we conclude that $(x-1)(x-4)\cdots(x-25)$ divides $f(x)$. Since $f(x)$ has degree at most 5,

$$f(x) = k(x-1)(x-4)(x-9)(x-16)(x-25)$$

for some constant k. However, equating

$$f(0) = [-(x+1)(x+2)(x+3)(x+4)(x+5)]_{x=0} = -5!$$

and $f(0) = -k\cdot 5!^2$ gives $k = \frac{1}{5!}$. Thus

$$f(x) = \frac{1}{5!}(x-1)(x-4)(x-9)(x-16)(x-25).$$

Then

$$\begin{aligned}\sum_{i=1}^{5}\frac{a_i}{6^2+i} &= \frac{f(6^2)}{x(x+1)(x+2)(x+3)(x+4)(x+5)|_{x=6^2}}-\frac{1}{6^2}\\ &= \frac{187465}{6744582}.\end{aligned}$$

5. Given two points (x_1, y_1) and (x_2, y_2) the equation

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \implies (y-y_1)(x_2-x_1)=(x-x_1)(y_2-y_1)$$

represents the line passing through these two points (it is a linear equation satisfied by the coordinares of the two points). It follows that three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on the same line if and only if the condition

$$(y_3-y_1)(x_2-x_1)=(x_3-x_1)(y_2-y_1) \tag{3}$$

holds. Now suppose that (x_1, y_1), (x_2, y_2) and (x_3, y_3) represent the (changing) coordinates of the three ducks as they waddle along their paths. Each coordinate is a linear function of time t, so (3) is an equation in t of degree at most 2 (i.e., is a quadratic). If such an equation has more than 2 solutions then it must reduce to an identity and thus hold true for all values of t. That is, if the ducks are in a row at more than two times, then they are always in a row.

6. Note that $g(x)$ is one of the 16 integer divisors of 2008 for each of the 81 integer roots. There must be at least 6 roots of $f(x)$ for which $g(x)$ has the same value. Since $g(x)$ is nonconstant, its degree must be greater than 5.

7. (Official solution) Let $p(x)$ be the monic real polynomial of degree n. If $n = 1$, then $p(r) = r + a$ for some real number a, and $p(x)$ is the average of x and $x + 2a$, each of which has 1 real root. Now we assume that $n > 1$. Let

$$g(x) = (x - 2)(x - 4) \cdots (x - 2(n - 1)).$$

The degree of $g(x)$ is $n - 1$. Consider the polynomials

$$q(x) = x^n - kg(x), r(x) = 2p(x) - q(x) = 2p(x) - x^n + kg(x).$$

We will show that for large enough k these two polynomials have n real roots. Since they are monic and their average is clearly $p(x)$, this will solve the problem.

Consider the values of the polynomial $g(x)$ at n points $x = 1, 3, 5, \ldots, 2n-1$. These values alternate in sign and are at least 1 (since at most two of the factors have magnitude 1 and the others have magnitude at least 2). On the other hand, there is a constant $c > 0$ such that for $0 \leq x \leq n$, we have $|x^n| < c$ and $|2p(x)-x^n| < c$. Take $k > c$. Then we see that $q(x)$ and $r(x)$ evaluated at n points $x = 1, 3, 5, \ldots, 2n - 1$ alternate in sign. Thus $q(x)$ and $r(x)$ each has at least $n - 1$ real roots. However since they are polynomials of degree n, they must have n real roots, as desired.

8. Without loss of generality, suppose $\deg(f) < \deg(g)$. Let $r_1, \ldots, r_k$ be the distinct roots of f and let $s_1, \ldots, s_l$ be the distinct roots of $f - 1$.

We claim that $k + l \geq \deg(n) + 1$. Indeed, suppose

$$f(x) = (x - r_1)^{p_1} \cdots (x - r_k)^{p_k}.$$

Then

$$(x - r_1)^{p_1-1} \cdots (x - r_k)^{p_k-1} \mid f'.$$

Similarly, if

$$f(x) - 1 = (x - s_1)^{q_1} \cdots (x - s_l)^{q_l},$$

then

$$(x - s_1)^{q_1-1} \cdots (x - s_l)^{q_l-1} \mid f'.$$

Since the roots of f and $f - 1$ are distinct,

$$(x - r_1)^{p_1-1} \cdots (x - r_k)^{p_k-1}(x - s_1)^{q_1-1} \cdots (x - s_l)^{q_l-1} \mid f'.$$

Since f' has degree $n - 1$,

$$(p_1 - 1) + \ldots + (p_k - 1) + (q_1 - 1) + \ldots (q_l - 1) \leq n - 1.$$

Since $p_1 + \ldots + p_k = q_1 + \ldots + q_l = n$, this gives $(n - k) + (n - l) \leq n - 1$, or $k + l \geq n + 1$.

Now $f-g$ has degree at most n and has at least $n+1$ distinct roots $r_1, r_2, \ldots, r_k, s_1, \ldots, s_l$, so it must be identically 0, and $f = g$.

2 Symmetric Polynomials and Vieta's Formulas

1. Note the polynomial has degree 2000 since the x^{2001} terms cancel out. By the Binomial Theorem, the coefficients of x^{2000} and x^{1999} are $2001\left(\frac{1}{2}\right)$ and $-\binom{2001}{2}\left(\frac{1}{2}\right)^2$, respectively. By Vieta's formula the sum of the roots is

$$-\frac{-\binom{2001}{2}\left(\frac{1}{2}\right)^2}{2001\left(\frac{1}{2}\right)} = 500.$$

2. Using Vieta's formulas with the roots r_i,

$$\begin{aligned}
\left(\sum \frac{1}{r_1^2}\right) &= \left(\sum \frac{1}{r_1}\right)^2 - 2\left(\sum \frac{1}{r_1 r_2}\right) \\
&= \left(\frac{\sum r_1 r_2 r_3 r_4}{r_1 r_2 r_3 r_4 r_5}\right)^2 - 2\left(\frac{\sum r_1 r_2 r_3}{r_1 r_2 r_3 r_4 r_5}\right) \\
&= \frac{9^2}{(-11)^2} - \frac{2(-7)}{-11} = -\frac{73}{121}
\end{aligned}$$

3. (a) The roots r_1, r_2, r_3 satisfy the equation $\left(\frac{1}{x}\right)^3 + a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0$. Clearing denominators, they are roots to $cx^3 + bx^2 + ax + 1$, and hence to

$$x^3 + \frac{b}{c}x^2 + \frac{a}{c}x + \frac{1}{c}.$$

   (b) By Vieta's formula, $r_1 + r_2 + r_3 = -a$, $r_1r_2 + r_2r_3 + r_3r_1 = b$, and $r_1r_2r_3 = -c$. We calculate the elementary symmetric sums in $r_1 + r_2, r_2 + r_3, r_3 + r_1$:

$$\begin{aligned}
(r_1 + r_2) + (r_2 + r_3) + (r_3 + r_1) &= -2a \\
(r_1 + r_2)(r_2 + r_3) + (r_2 + r_3)(r_3 + r_1) & \\
+(r_3 + r_1)(r_1 + r_2) &= (r_1 + r_2 + r_3)^2 + (r_1r_2 + r_2r_3 + r_3r_1) \\
&= a^2 + b \\
(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) &= (r_1 + r_2 + r_3)(r_1r_2 + r_2r_3 + r_3r_1) - r_1r_2r_3 \\
&= -ab + c
\end{aligned}$$

   Hence by Vieta's formulas (using the roots to get the coefficients), $r_1 + r_2, r_2 + r_3, r_3 + r_1$ are roots of

$$x^3 + 2ax^2 + (a^2 + b)x + (ab - c).$$

   (c) We calculate the elementary symmetric sums in r_1^2, r_2^2, r_3^2:

$$\begin{aligned}
r_1^2 + r_2^2 + r_3^2 &= (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_2r_3 + r_3r_1) = a^2 - 2b \\
r_1^2r_2^2 + r_2^2r_3^2 + r_3^2r_1^2 &= (r_1r_2 + r_2r_3 + r_3r_1)^2 - 2r_1r_2r_3(r_1 + r_2 + r_3) = b^2 - 2ac \\
r_1^2r_2^2r_3^2 &= c^2
\end{aligned}$$

   Hence by Vieta's formulas, r_1^2, r_2^2, r_3^2 are roots of

$$x^3 + (2b - a^2)x^2 + (b^2 - ac)x - c^2.$$

4. The coefficient of x^2 is 0 so $r+s+t=0$. Using $rst = \frac{-2008}{8}$, we get

$$\begin{aligned}(r+s)^3+(s+t)^3+(t+r)^3 &= 2(r+s+t)^3 - 3(r+s+t)(rs+st+tr) - 3rst \\ &= -3\left(\frac{-2008}{8}\right) = -753.\end{aligned}$$

5. Let $y = 2^{111x}$; the equation becomes $\frac{1}{4}y^3 + 4y = 2y^2 + 1$ which rearranges to $y^3 - 8y^2 + 16y - 4 = 0$. Let y_1, y_2, y_3 be the roots of this equation and x_1, x_2, x_3 be the solutions to the original equation. Then

$$2^{111(x_1+x_2+x_3)} = y_1y_2y_3 = 4$$

by Vieta's formula so $x_1+x_2+x_3 = \frac{1}{111}\log_2 4 = \frac{2}{111}$ and the answer is 113.

6. To simplify the calculation, we first divide $P(x)$ by $Q(x)$ to obtain

$$P(x) = Q(x)(x^2+1) + x^2 - x + 1.$$

Thus

$$\sum_{i=1}^{4} P(z_i) = \sum_{i=1}^{4} Q(z_i)(z_i^2+1) + \sum_{i=1}^{4}(z_i^2 - z_i + 4) = \sum_{i=1}^{4}(z_i^2 - z_i + 4).$$

The first and second elementary symmetric sums equal 1 and -1 by Vieta. Hence the above sum equals

$$\left(\sum_{i=1}^{4} z_i\right)^2 - 2\sum_{1\le i<j\le 4} z_iz_j - \sum_{i=1}^{4} z_i + 4 = 1+2-1+4 = 6.$$

3 Fundamental Theorem of Algebra

1. Write the equation as

$$(g(x)+ih(x))(g(x)-ih(x)) = \frac{x^{20}-1}{x^2-1}.$$

Counting the number of possibilities for $(f(x), g(x))$ is the same as counting the number of possibilities for $f(x) = g(x) + ih(x)$. Thus we need to count the number of complex polynomials $f(x)$ such that

$$f(x)\bar{f}(x) = \frac{x^{20}-1}{x^2-1}.$$

The zeros of $\frac{x^{20}-1}{x^2-1}$ can be split in complex conjugate pairs $P_1, \ldots, P_9$, since they are the nonreal 20th roots of unity. If r is a zero of $f(x)$ then $\bar{r}$ is a zero of $\bar{f}(x)$. Thus $f(x)$ must have as a zero one number in each pair P_i, and $\bar{f}(x)$ has as its zeros the other number in each pair P_i. There are $2^9 = 512$ choices for which zeros in each pair to choose as zeros of $f(x)$. The answer is 512.